STAT 712 hw 1

Set theory, probability axioms, counting

Do problems 1.2, 1.3, 1.8, 1.10, 1.11, 1.12, 1.13, 1.14, 1.18, 1.19, 1.23 from CB. In addition:

1. For a collection of sets $A_1, \ldots, A_n$, $n \geq 2$, we have

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum\sum_{1 \leq i_1 < i_2 \leq n} P(A_{i_1} \cap A_{i_2}) + \sum\sum\sum_{1 \leq i_1 < i_2 < i_3 \leq n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n+1} P(\cap_{i=1}^n A_i).$$

This is known as the *inclusion-exclusion* principle.

(a) Prove this result by induction. That is, prove it for $n = 2$ and then show that if it is true for an arbitrary $n \geq 2$, it must be true for $n + 1$.

We have proven the base case in class: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Now, assuming the formula holds for n, we write

$$\begin{aligned} P(\cup_{i=1}^{n+1} A_i) &= P(\cup_{i=1}^n A_i \cup A_{n+1}) \\ &= P(\cup_{i=1}^n A_i) + P(A_{n+1}) - P((\cup_{i=1}^n A_i) \cap A_{n+1}) \\ &= P(\cup_{i=1}^n A_i) + P(A_{n+1}) - P(\cup_{i=1}^n (A_i \cap A_{n+1})), \end{aligned}$$

using the $n = 2$ result. Now we apply the inclusion-exclusion formula for an arbitrary n to the first and the third term of the above. This gives

$$
\begin{aligned}
P(\cup_{i=1}^{n+1} A_i) = & \sum_{i=1}^{n} P(A_i) - \sum\sum_{1\le i_1<i_2\le n} P(A_{i_1}\cap A_{i_2}) + \sum\sum\sum_{1\le i_1<i_2<i_3\le n} P(A_{i_1}\cap A_{i_2}\cap A_{i_3}) - \dots \\
& + (-1)^{n+1} P(\cap_{i=1}^{n} A_i) + P(A_{n+1}) \\
& - \left[\sum_{i=1}^{n} P(A_i \cap A_{n+1}) - \sum\sum_{1\le i_1<i_2\le n+1} P(A_{i_1}\cap A_{i_2}\cap A_{n+1}) \right. \\
& \quad + \sum\sum\sum_{1\le i_1<i_2<i_3\le n} P(A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{n+1}) \\
& \left. +(-1)^n \sum_{1\le i_1<\dots<i_{n-1}\le n} \cdots \sum P(A_{i_1}\cap\cdots\cap A_{i_{n-1}}\cap A_{n+1}) + (-1)^{n+1} P(\cap_{i=1}^{n}(A_i\cap A_{n+1})) \right] \\
= & \left[\sum_{i=1}^{n} P(A_i) + P(A_{n+1}) \right] - \left[\sum\sum_{1\le i_1<i_2\le n} P(A_{i_1}\cap A_{i_2}) - \sum_{i=1}^{n} P(A_i\cap A_{n+1}) \right] \\
& + \left[\sum\sum\sum_{1\le i_1<i_2<i_3\le n} P(A_{i_1}\cap A_{i_2}\cap A_{i_3}) + \sum\sum_{1\le i_1<i_2\le n+1} P(A_{i_1}\cap A_{i_2}\cap A_{n+1}) \right] - \\
& \cdots + \left[(-1)^n \sum_{1\le i_1<\dots<i_{n-1}\le n} \cdots \sum P(A_{i_1}\cap\cdots\cap A_{i_{n-1}}\cap A_{n+1}) + (-1)^{n+1} P(\cap_{i=1}^{n} A_i) \right] \\
& + (-1)^{n+2} P(\cap_{i=1}^{n+1} A_i)
\end{aligned}
$$

We see that we can match terms as in the sets of square brackets above and simplify. This gives the result:

$$
\begin{aligned}
P(\cup_{i=1}^{n+1} A_i) = & \sum_{i=1}^{n+1} P(A_i) - \sum\sum_{1\le i_1<i_2\le n+1} P(A_{i_1}\cap A_{i_2}) \\
& + \sum\sum\sum_{1\le i_1<i_2<i_3\le n+1} P(A_{i_1}\cap A_{i_2}\cap A_{i_3}) - \cdots + (-1)^{n+2} P(\cap_{i=1}^{n+1} A_i).
\end{aligned}
$$

(b) Suppose n guests take n seats around a table at random. Then the host rearranges them according to a seating chart he made before the guests' arrival.

i. Find the probability that every guest must move to a different seat. *Hint: Let A_i be the event that guest i sits in his or her assigned seat for $i = 1, \dots, n$.*

Letting A_i be the event that guest i sits in his or her assigned seat for $i = 1, \dots, n$, the event that every guest must move to a different seat is the event $\cap_{i=1}^{n} A_i^c$, which by De Morgan's Laws is the event $(\cup_{i=1}^{n} A_i)^c$. We can find $P(\cup_{i=1}^{n} A_i)$ using the inclusion-exclusion principle and subtract this from 1 to get the final answer. Noting that there are $n!$ possible seating arrangements, we have

$$\begin{aligned}
P(A_i) &= (n-1)!/n! \quad \text{(Put guest } i \text{ in correct seat; arrange others.)} \\
P(A_i \cap A_j) &= (n-2)!/n! \quad \text{(Put guests } i \text{ and } j \text{ in correct seats; arrange others.)} \\
&\vdots \\
P(A_{i_1} \cap \cdots \cap A_{i_m}) &= (n-m)!/n! \quad \text{(Put guests } i_1, \ldots, i_m \text{ in correct seats; arrange others.)}
\end{aligned}$$

So we have

$$\begin{aligned}
P(\cup_{i=1}^n A_i) &= \sum_{i=1}^n \frac{(n-1)!}{n!} - \sum_{1 \le i_1 < i_2 \le n} \frac{(n-2)!}{n!} + \sum_{1 \le i_1 < i_2 < i_3 \le n} \frac{(n-3)!}{n!} - \cdots + (-1)^{n+1} \frac{1}{n!} \\
&= n \frac{(n-1)!}{n!} - \binom{n}{2} \frac{(n-2)!}{n!} + \binom{n}{3} \frac{(n-3)!}{n!} - \cdots + (-1)^{n+1} \frac{1}{n!} \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!} \\
&= \sum_{i=1}^n (-1)^{i+1} \frac{1}{i!}
\end{aligned}$$

So the answer is

$$P(\cap_{i=1}^n A_i^c) = 1 - \sum_{i=1}^n (-1)^{i+1} \frac{1}{i!}.$$

ii. Find the limit of this probability as $n \to \infty$.

We find that

$$\lim_{n \to \infty} \sum_{i=1}^n (-1)^{i+1} \frac{1}{i!} = 1 - e^{-1}$$

by considering the Taylor expansion of the function e^{-x} around $x = 0$. So we have

$$\lim_{n \to \infty} P(\cap_{i=1}^n A_i^c) = 1 - \lim_{n \to \infty} \sum_{i=1}^n (-1)^{i+1} \frac{1}{i!} = e^{-1} = 0.3678794.$$

(c) Two 52-card decks are shuffled and place side-by-side. From each deck a card is drawn and placed face-up. This is repeated 52 times, resulting in 52 pairs of cards drawn. What is the probability that at least one pair is a match?

Letting A_i be the event that the ith pair is a match, we have

$$\begin{aligned}
P(A_i) &= \frac{52 \cdot 51! \cdot 51!}{52! \cdot 52!} = \frac{1}{52} \\
P(A_i \cap A_j) &= \frac{52 \cdot 51 \cdot 50! \cdot 50!}{52! \cdot 52!} = \frac{(52-2)!}{52!} \\
P(A_i \cap A_j \cap A_k) &= \frac{52 \cdot 51 \cdot 50 \cdot 49! \cdot 49!}{52! \cdot 52!} = \frac{(52-3)!}{52!} \\
&\vdots \\
P(A_{i_1} \cap \cdots \cap A_{i_m}) &= \frac{(52-m)!}{52!}.
\end{aligned}$$

To obtain the expression for $P(A_i)$, we divide the number of ways in which flip i can be a match by the total number of ways in which the cards in the two decks can be arranged. The latter is $52! \cdot 52!$. The former is obtained by considering that to arrange for a match on the ith flip, we can place any of the 52 cards in position i of the deck (52 ways to do this), rearranging the rest of the cards in any way ($51!$, and then placing the same card of the other deck in position i and arranging the rest of the cards in any way ($51!$ ways to do this). So the numerator is $52 \cdot 51! \cdot 51!$.

We obtain the expression for $P(A_i \cap A_j)$ similarly: There are 52 ways to choose a card to place in position i of the first deck and then 51 ways to choose a card to place in position j of the first deck. Then there are $50!$ ways to arrange in rest of the cards. Then after placing the same cards in positions i and j in the second deck, there are $50!$ ways to arrange the rest of the cards in the second deck. So the numerator is $52 \cdot 51 \cdot 50! \cdot 50!$

Now, by the inclusion-exclusion principle, we have

$$\begin{aligned}
P(\cup_{i=1}^{52} A_i) &= \sum_{m=1}^{52} (-1)^{m+1} \binom{52}{m} \frac{(52-m)!}{52!} \\
&= \sum_{m=1}^{52} (-1)^{m+1} \frac{1}{m!} \\
&= 0.6321206.
\end{aligned}$$

Problems 1.3, 1.11, 1.12(b), 1.14, 1.18, 1.23 from CB.

$\boxed{1.3}$ (a) Proof of $A \cup B = B \cup A$: We have

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B \Rightarrow x \in B \cup A,$$

so $A \cup B \subset B \cup A$. Also

$$x \in B \cup A \Rightarrow x \in B \text{ or } x \in A \Rightarrow x \in A \cup B,$$

so $B \cup A \subset A \cup B$. Therefore

$$A \cup B = B \cup A.$$

Proof of $A \cap B = B \cap A$: We have

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B \Rightarrow x \in B \cap A$$

so $A \cap B \subset B \cap A$. Also,

$$x \in B \cap A \Rightarrow x \in B \text{ and } x \in A \Rightarrow x \in A \cap B$$

so $B \cap A \subset A \cap B$. Therefore

$$A \cap B = B \cap A.$$

(b) Proof of $A \cup (B \cup C) = (A \cup B) \cup C$: We have

$$\begin{aligned} x \in A \cup (B \cup C) &\Rightarrow x \in A \text{ or } x \in B \cup C \\ &\Rightarrow x \in A \text{ or } x \in B \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \text{ or } x \in C \end{aligned}$$

$$\Rightarrow x \in (A \cup B) \cup C,$$

so $A \cup (B \cup C) \subset (A \cup B) \cup C$. Also

$$x \in (A \cup B) \cup C \Rightarrow x \in A \cup B \text{ or } x \in C$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ or } x \in C$$

$$\Rightarrow x \in A \text{ or } x \in (B \cup C)$$

$$\Rightarrow x \in A \cup (B \cup C),$$

so $(A \cup B) \cup C \subset A \cup (B \cup C)$. Therefore

$$(A \cup B) \cup C = A \cup (B \cup C).$$

<u>Proof of $A \cap (B \cap C) = (A \cap B) \cap C$</u>: We have

$$x \in A \cap (B \cap C) \Rightarrow x \in A \text{ and } x \in (B \cap C)$$

$$\Rightarrow x \in A \text{ and } x \in B \text{ and } x \in C$$

$$\Rightarrow x \in (A \cap B) \text{ and } x \in C$$

$$\Rightarrow x \in (A \cap B) \cap C,$$

so $A \cap (B \cap C) \subset (A \cap B) \cap C$. Also

$$x \in (A \cap B) \cap C \Rightarrow x \in (A \cap B) \text{ and } x \in C$$

$$\Rightarrow x \in A \text{ and } x \in B \text{ and } x \in C$$

$\Rightarrow \quad x \in A \quad$ and $\quad x \in (B \cap C)$

$\Rightarrow \quad x \in A \cap (B \cap C)$,

so $\quad (A \cap B) \cap C \subset A \cap (B \cap C)$. Therefore

$$(A \cap B) \cap C = A \cap (B \cap C).$$

(c) <u>Proof of $(A \cup B)^c = A^c \cap B^c$</u>: We have

$x \in (A \cup B)^c \quad \Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \notin A \quad$ and $\quad x \notin B$

$\Rightarrow \quad x \in A^c \quad$ and $\quad x \in B^c$

$\Rightarrow \quad x \in A^c \cap B^c$,

so $\quad (A \cup B)^c = A^c \cap B^c$. Also

$x \in A^c \cap B^c \quad \Rightarrow \quad x \in A^c \quad$ and $\quad x \in B^c$

$\Rightarrow \quad x \notin A \quad$ and $\quad x \notin B$

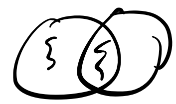
$\Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \in (A \cup B)^c$,

so $\quad A^c \cap B^c \subset (A \cup B)^c$. Therefore

$$(A \cup B)^c = A^c \cap B^c.$$

<u>Proof of $(A \cap B)^c = A^c \cup B^c$:</u> We have



$$x \in (A \cap B)^c \Rightarrow x \notin A \cap B$$

$$\Rightarrow x \in A^c \text{ or } x \in B^c$$

$$\Rightarrow x \in A^c \cup B^c$$

so $(A \cap B)^c \subset A^c \cup B^c$. Also

$$x \in A^c \cup B^c \Rightarrow x \in A^c \text{ or } x \in B^c$$

$$\Rightarrow x \notin A \cap B$$

$$\Rightarrow x \in (A \cap B)^c,$$

so $A^c \cup B^c \Rightarrow (A \cap B)^c$. Therefore

$$(A \cap B)^c = A^c \cup B^c.$$

Here is the proof of the distributive laws too:

<u>Proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:</u> We have

$$x \in A \cap (B \cup C) \Rightarrow x \in A \text{ and } x \in B \cup C.$$

$$\Rightarrow x \in A \text{ and } x \in B$$

$$\text{or } x \in A \text{ and } x \in C.$$

$$\Rightarrow \quad x \in A \cap B \quad \text{or} \quad x \in A \cap C$$

$$\Rightarrow \quad x \in (A \cap B) \cup (A \cap C),$$

so $\quad A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Also,

$$x \in (A \cap B) \cup (A \cap C) \quad \Rightarrow \quad x \in A \cap B \quad \text{or} \quad x \in A \cap C$$

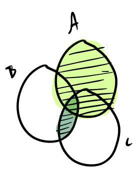
$$\Rightarrow \quad x \in A \quad \text{and} \quad x \in B \cup C$$

$$\Rightarrow \quad x \in A \cup (B \cup C),$$

so $\quad (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

<u>Proof of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:</u> We have

$$x \in A \cup (B \cap C) \quad \Rightarrow \quad x \in A \quad \text{or} \quad x \in B \cap C$$



$$\Rightarrow \quad x \in A \cup B \quad \text{and} \quad x \in A \cup C$$

$$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C),$$

so $\quad A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Also

$$x \in (A \cup B) \cap (A \cup C) \quad \Rightarrow \quad x \in A \quad \text{or} \quad x \in B \cap C$$

$$\Rightarrow \quad x \in A \quad \text{or} \quad x \in (B \cap C)$$

$$\Rightarrow \quad x \in A \cup (B \cap C),$$

so $\quad (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Therefore

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$\boxed{1.11}$ Let S be a sample space.

(a) Show that $\mathcal{B} = \{\emptyset, S\}$ is a sigma-algebra.

Recall: $\mathcal{B}$ is a σ-algebra if

1. $\emptyset \in \mathcal{B}$
2. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$
3. $A_1, A_2, \ldots \in \mathcal{B} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

We have

1. $\emptyset \in \mathcal{B}$,
2. $\emptyset^c = S \in \mathcal{B}$, $S^c = \emptyset \in \mathcal{B}$, and
3. $\emptyset \cup S = S \in \mathcal{B}$,

so $\mathcal{B}$ <u>is</u> a σ-algebra.

(b) Let $\mathcal{B} = \{$all subsets of S including S itself$\}$

We have
1. $\emptyset$ is the subset of S containing no elements of S, so $\emptyset \in \mathcal{B}$.

2. For $A \in \mathcal{B}$, $A \subset S$.
   Then $A^c = S \setminus A \subset S$, so $A^c \in \mathcal{B}$.

3. For $A_1, A_2, \ldots \in \mathcal{B}$, $A_1, A_2, \ldots \subset S$, so $\bigcup_{i=1}^{\infty} A_i \subset S$,
   this means $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

(c) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be σ-algebras on S.

We have

1. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are both σ-algebras, $\emptyset \in \mathcal{B}_1$ and $\emptyset \in \mathcal{B}_2$, so

$$\emptyset \in \mathcal{B}_1 \cap \mathcal{B}_2.$$

2. Let $A \in \mathcal{B}_1 \cap \mathcal{B}_2$. Then $A \in \mathcal{B}_1$ and $A \in \mathcal{B}_2$.

   Then $A^c \in \mathcal{B}_1$ and $A^c \in \mathcal{B}_2$, so

$$A^c \in \mathcal{B}_1 \cap \mathcal{B}_c.$$

3. Let $A_1, A_2, \ldots \in \mathcal{B}_1 \cap \mathcal{B}_2$.

   Then $A_1, A_2, \ldots \in \mathcal{B}_1$ and $A_1, A_2, \ldots \in \mathcal{B}_2$.

   therefore $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_1$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_2$, so

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_1 \cap \mathcal{B}_2.$$

So $\mathcal{B}_1 \cap \mathcal{B}_2$ <u>is</u> a σ-algebra.

$\boxed{1.12}$ (b) Show that finite additivity & continuity $\Rightarrow$ countable additivity

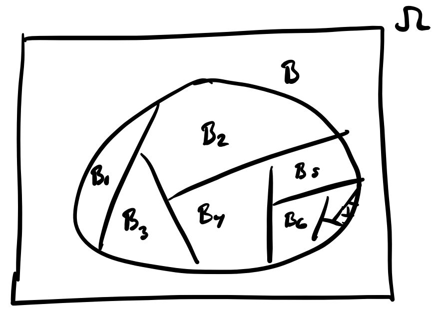
Assume (continuity):

IF $A_n \supset A_{n+1} \supset \cdots$ is a decreasing seq. s.t. $\lim_{n\to\infty} A_n = \emptyset$ then $\lim_{n\to\infty} P(A_n) = 0$.

Let $B_1, B_2, \ldots$ be pwise disjoint. We want to show $P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$.

Define $B = \bigcup_{n=1}^{\infty} B_n$

and $\underbrace{R_n}_{\text{"remainder"}} = B \setminus \left(\bigcup_{i=1}^{n} B_i\right)$ for $n \geq 1$, noting that $R_n \supset R_{n+1} \supset \cdots$.



Then $B = \left(\bigcup_{i=1}^{n} B_i\right) \cup R_n$

disjoint

and

$$P(B) = \sum_{i=1}^{n} P(B_i) + P(R_n) \quad \forall\ n \geq 1$$

by finite additivity. Now if $\lim_{n\to\infty} R_n = 0$ then $\lim_{n\to\infty} P(R_n) = 0$, so that

$$P(B) = \lim_{n\to\infty} \left[\sum_{i=1}^{n} P(B_i) + P(R_n)\right] = \sum_{n=1}^{\infty} P(B_n),$$

which is the axiom of countable additivity.

Now let's show $\lim_{n\to\infty} R_n = \emptyset$. We have

$$\limsup_{n\to\infty} R_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} R_k$$

$$= \bigcap_{n=1}^{\infty} R_n$$

$$= \bigcap_{n=1}^{\infty} B \setminus \left(\bigcup_{i=1}^{n} B_i \right)$$

$$= \bigcap_{n=1}^{\infty} B \cap \left(\bigcup_{i=1}^{n} B_i \right)^c$$

$$= \bigcap_{n=1}^{\infty} B \cap \left(\bigcap_{i=1}^{n} B_i^c \right)$$

$$= B \cap \left(\bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} B_i^c \right)$$

$$= B \cap \{ \omega \in \Omega : \forall\, n \geq 1 \quad \omega \in B_i^c \quad \forall\, i \leq n \}$$

$$= B \cap \{ \omega \in \Omega : \forall\, n \geq 1 \quad \omega \in B_i^c \}$$

$$= B \cap \left(\bigcap_{n=1}^{\infty} B_n^c \right)$$

$$= B \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^c$$

$$= B \cap B^c$$

$$= \emptyset.$$

$$\liminf_{n \to \infty} R_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} R_k = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} R_k = \bigcup_{n=1}^{\infty} \emptyset = \emptyset.$$

$R_n \supset R_{n+1} \supset \cdots$ work above

$\boxed{1.14}$ # of subsets that can be made from sample space of n sample points is 2^n.

For each of the n sample points, decide whether to include it or not: n tasks, 2 ways to do each one. The job can be done in 2^n ways.

$\boxed{1.18}$ Place n balls into n cells. What is the probability that exactly 1 cell empty?

$$\underbrace{\circ \ \circ \ \circ \ \cdots \ \circ}_{n \text{ balls}}$$

$$\underbrace{\_\ \_\ \_\ \cdots\ \_}_{n \text{ cells}}$$

For each of the n balls, you can choose any of n cells: Job can be done in n^n ways

If one cell empty, then of the remaining cells, one has two balls and the rest have one ball.

tasks	# ways
choose empty cell	n
choose cell to have two balls	$(n-1)$
choose two balls to place in double cell	$\binom{n}{2}$
choose ball for 1^{st} remaining cell	$n-2$
2^{nd}	
$\vdots$	
$(n-2)^{th}$	1

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

ways to do job $\quad n \cdot (n-1) \cdot \binom{n}{2} (n-2) \cdot \ldots \cdot 1 = \binom{n}{2} n!$.

$$P(\text{exactly 1 cell empty}) = \frac{\binom{n}{2} n!}{n^n}$$

$\boxed{1.23}$ Two people toss coin n times. Find the probability that they get the same # of heads.

sequences of n coin tosses: $2^n \cdot 2^n = 4^n$.

ways each get i heads $\binom{n}{i}\binom{n}{i}$.

So # ways to get same # heads is

$$\sum_{i=0}^{n} \binom{n}{i}^2 .$$

fun identity

$$P(\text{same number heads}) = \frac{\sum_{i=0}^{n} \binom{n}{i}^2}{4^n} = \left(\frac{1}{4}\right)^n \binom{2n}{n}$$

Can use the cool identity $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$, which we can see by writing

$$(1+x)^n (1+x)^n = \left(\sum_{i=0}^{n} x^i \binom{n}{i}\right)\left(\sum_{j=0}^{n} x^j \binom{n}{j}\right)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n} x^i x^j \binom{n}{i}\binom{n}{j}$$

$$= \cdots + \underbrace{x^n \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}}_{x^n \text{ term in polynomial}} + \cdots$$

$$(1+x)^{2n} = \sum_{i=0}^{n} x^i \binom{2n}{i} = \cdots + \underbrace{x^n \binom{2n}{n}}_{x^n \text{ term in polynomial.}} + \cdots$$

$$\text{So} \quad \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\underbrace{\binom{n}{n-i}}_{=\binom{n}{i}} = \sum_{i=0}^{n} \binom{n}{i}^2 .$$