STAT 712 hw 2

Basic probability, conditional probability, Bayes' rule, cdfs, pdfs, pmfs

Do problems 1.27, 1.33, 1.34, 1.35, 1.36, 1.38, 1.39, 1.49, 1.51, 1.52, 1.53, 1.54 from CB. In addition:

1. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of sets. Show that $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$.

We have

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{=A_n} = \bigcap_{n=1}^{\infty} A_n.$$

Also

$$\begin{aligned}\liminf_{n\to\infty} A_n &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} \{\omega : \omega \in A_k \ \forall k \geq n\} \\ &= \{\omega : \omega \in A_k \ \forall k \geq 1\} \\ &= \bigcap_{k=1}^{\infty} A_k.\end{aligned}$$

Since $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ exist and are equal to $\bigcap_{n=1}^{\infty} A_n$, we have $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

Problems 1.38, 1.39, 1.49, 1.54 from CB

$\boxed{1.38}$ Prove the following:

(a) If $P(B)=1$ then $P(A|B)=P(A)$ for any A.

We have

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \underset{P(A)=P(A\cap B)+P(A\cap B^c)}{P(A\cap B) = P(A) - \overbrace{P(A\cap B^c)}^{=0}} = P(A),$$

where $P(A\cap B^c)=0$ because $A\cap B^c \subset B^c$ and $P(B^c)=0$, giving

$$0 \le P(A\cap B^c) \le P(B^c) = 0.$$

(b) If $A\subset B$ then $P(B|A)=1$ and $P(A|B)=P(A)/P(B)$.

Since $A\subset B \Rightarrow A = A\cap B$, we have

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

Also

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

(c) If $A\cap B=\emptyset$ then $P(A|A\cup B) = \dfrac{P(A)}{P(A)+P(B)}$.

We have $P(A\cup B) = P(A)+P(B) - \underbrace{P(A\cap B)}_{=0} = P(A)+P(B)$. So

$$P(A|A\cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{P((A\cap A)\cup(A\cap B))}{P(A)+P(B)} = \frac{P(A)}{P(A)+P(B)}.$$

(d) $P(A \cap B \cap C) = P(A | B \cap C)\, P(B|C)\, P(C)$.

We have

$$P(A| B\cap C) = \frac{P(A \cap B \cap C)}{P(B\cap C)} \quad , \quad P(B|C) = \frac{P(B\cap C)}{P(C)}.$$

So

$$P(A | B\cap C) = \frac{P(A\cap B \cap C)}{P(B|C)\, P(C)},$$

which we can rearrange to obtain the result.

$\boxed{1.39}$ If $P(A) > 0$ and $P(B) > 0$

(a) $A \cap B = \emptyset \Rightarrow A, B$ not indep.

Suppose A, B indep. Then $P(A\cap B) = P(A)\,P(B)$.

But $P(A\cap B) = P(\emptyset) = 0$ and $P(A) P(B) > 0$

so there is a contradiction.

Therefore A, B are not independent.

(b) $P(A\cap B) = P(A)\, P(B) \Rightarrow A\cap B \neq \emptyset$.

Suppose $A\cap B = \emptyset$. Then $P(A\cap B) = 0$.

But $P(A\cap B) = P(A)\, P(B) > 0$, so there is a contradiction.

Therefore $A \cap B \neq \emptyset$, i.e. A, B are not mutually exclusive.

$\boxed{1.49}$ Suppose $X \sim F_X$, $Y \sim F_Y$ and $F_X(t) \le F_Y(t)$ $\forall\, t \in \mathbb{R}$

and $F_X(t) < F_Y(t)$ for some $t \in \mathbb{R}$.

Show $P(X > t) \underset{>}{\geq} P(Y > t)$ $\underset{\text{for some}}{\forall}$ $t \in \mathbb{R}$.

We have

$$P(X > t) = 1 - P(X \le t)$$

$$= 1 - F_X(t)$$

$\geq$ for all t, $>$ for some t

$$\geq 1 - F_Y(t)$$

$$= 1 - P(Y \le t)$$

$$= P(Y > t)$$

$\boxed{1.54}$

(a) $$1 = \int_0^{\pi/2} c \cdot \sin x \, dx = c \cdot \left[-\cos x\right]\Big|_0^{\pi/2} = c \cdot \left(-0 - (-1)\right) = c$$

So $c = 1$.

(b) $$1 = \int_{-\infty}^{\infty} c \cdot e^{-|x|} \, dx$$

$$= 2 \cdot c \cdot \int_0^{\infty} e^{-x} \, dx$$

$$= 2 \cdot c \cdot [-e^{-x}]\Big|_0^{\infty}$$

$$= 2c$$

so $c = 1/2$.