STAT 712 hw 3

Expected value, variance, mgfs

Do problems 2.14, 2.17, 2.24, 2.26, 2.28, 2.32, 2.38 from CB. In addition:

1. Each of N visitors entering a museum must pass through one of n turnstiles. Suppose each visitor chooses a turnstile at random, independently of the others, and let X_n be the number of visitors that enter through turnstile n.

   (a) Give the pmf of X_n.

   We have
   $$p_X(x) = \binom{N}{x}\left(\frac{1}{n}\right)^x\left(1-\frac{1}{n}\right)^{N-x}, \quad x = 0, 1, \dots, N.$$

   (b) Give the mgf M_{X_n} of X_n.

   We see that $X_n \sim \text{Binomial}(N, 1/n)$, so
   $$M_{X_n}(t) = [e^t(1/n) + (1-1/n)]^N.$$

   (c) For a positive integer k, let $N = k \cdot n$, so that k visitors per turnstile enter, and find $\lim_{n\to\infty} M_{X_n}$.

   With $N = kn$, we have $X_n \sim \text{Binomial}(kn, 1/n)$, which has mgf given by
   $$M_{X_n}(t) = [e^t(1/n) + (1-1/n)]^{kn} = \left([1 + (e^t - 1)/n]^n\right)^k \to e^{k(e^t-1)}$$
   as $n \to \infty$.

   (d) Give the limiting distribution of X_n as $n \to \infty$ when $N = k \cdot n$.

   The mgf of X_n converges to that of the Poisson(k) distribution for all $t \in \mathbb{R}$, so the limiting distribution of X_n is the Poisson(k) distribution.

2. (Optional) A rv X is called b-sub-Gaussian if for some $b > 0$, $\mathbb{E}e^{tX} \le e^{b^2t^2/2}$ for all $t \in \mathbb{R}$.

   (a) Show that if X is b-sub-Gaussian, then
   $$P(|X| \ge a) \le 2e^{-a^2/(2b^2)} \quad \text{for all } a > 0.$$

For any $t > 0$, we may write

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{\mathbb{E}e^{tX}}{e^{ta}} \leq \frac{e^{b^2t^2/2}}{e^{ta}} = e^{-ta+b^2t^2/2},$$

where the first inequality comes from Markov's inequality. Since this bound holds for all $t > 0$, we can get the tightest bound by minimizing the right hand side in t. Doing this results in

$$P(X \geq a) \leq e^{-a^2/(2b^2)}.$$

Now we must consider $P(X \leq -a) = P(-X \geq a)$. For this we can obtain the same upper bound by noting that $\mathbb{E}e^{(-t)X} \leq e^{b^2t^2/2}$. Finally, we have

$$P(|X| \geq a) = P(\{X \geq a\} \cup \{X \leq -a\}) = P(\{X \geq a\}) + P(\{X \leq -a\}) \leq 2e^{-a^2/(2b^2)}.$$

(b) Show that $X \sim \text{Normal}(0, \sigma^2)$ is a σ-sub-Gaussian rv.

We have $\mathbb{E}e^{tX} = e^{\sigma^2t^2/2}$, so X satisfies the condition of σ-sub-Gaussianity.

(c) Show that $X \sim \text{Uniform}(-b, b)$ is a b-sub-Gaussian rv.
Hint: Write the mgf of X as an infinite series and use the inequality $(2k+1)! \geq k!2^k$.

We have

$$\begin{aligned}
\mathbb{E}e^{tX} &= \int_{-b}^{b} e^{tx}\frac{1}{2b}dx \\
&= \frac{1}{2bt}[e^{tb} - e^{-tb}] \\
&= \frac{1}{bt}\left[bt + \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} + \dots\right] \\
&= \sum_{k=0}^{\infty} \frac{(bt)^{2k}}{(2k+1)!} \\
&\leq \sum_{k=0}^{\infty} \frac{(bt)^{2k}}{k!2^k} \\
&= \sum_{k=0}^{\infty} \frac{(b^2t^2/2)^k}{k!} \\
&= e^{b^2t^2/2}.
\end{aligned}$$

Problems 2.14, 2.26, 2.28 (a), 2.32, 2.38 from CB

$\boxed{2.14}$ (a) For X nonnegative and continuous, show that

$$\mathbb{E}X = \int_0^\infty [1 - F_X(x)]\,dx.$$

We have

$$\int_0^\infty [1 - F_X(x)]\,dx = \int_0^\infty \left[1 - \int_0^x f_X(t)\,dt\right] dx$$

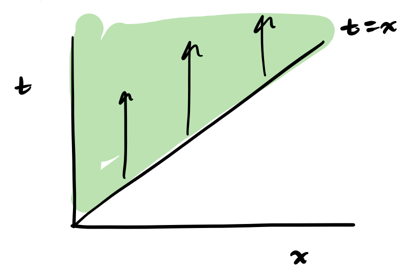
$$= \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx$$

$$= \int_0^\infty \int_0^t f_X(t)\,dx\,dt$$

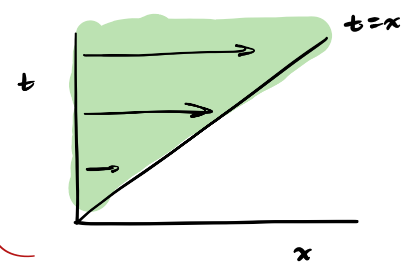
$$= \int_0^\infty t\, f_X(t)\,dt$$

$$= \mathbb{E}X.$$

For each $x \geq 0$, integrate in t from x to ∞:



For each $t \geq 0$ integrate in x from 0 to t:



(b) For X nonnegative, discrete with $\mathcal{X} = \{0, 1, 2, \ldots\}$. Show that

$$\mathbb{E}X = \sum_{x=0}^{\infty} [1 - F_X(x)].$$

We have

$$\sum_{x=0}^{\infty} [1 - F_X(x)] = \sum_{x=0}^{\infty} \left[1 - \sum_{i=0}^{x} p_X(i)\right]$$

$$= \sum_{x=0}^{\infty} \sum_{i=x+1}^{\infty} p_X(i)$$

$$= \sum_{i=1}^{\infty} \sum_{x=0}^{i-1} p_X(i)$$

For each $x = 0, 1, 2,$ sum over i from $x+1$ to infinity.

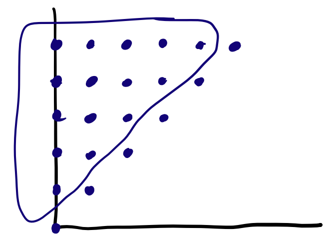
For each $i = 1, 2, 3, \ldots,$ sum over x from 0 to $i-1$

$$= \sum_{i=1}^{\infty} i \; p_X(i)$$

(since $0 \cdot p_X(0) = 0$)

$$= \sum_{i=0}^{\infty} i \; p_X(i)$$

$$= \mathbb{E}X.$$



$\boxed{2.26}$ Let $X \sim f_X$, where $f_X(a+\varepsilon) = f_X(a-\varepsilon) \;\; \forall \; \varepsilon > 0$.

(b) Show $\operatorname{med}(X) = a$

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\, dx$$

$$P_X(X \le a) = \int_{-\infty}^{a} f_X(x)\, dx$$

Let $x = a + t \iff t = x - a$,
so $x \in (-\infty, a) \iff t \in (-\infty, 0)$

$$= \int_{-\infty}^{0} f_X(a+t)\, dt$$

$$= \int_{-\infty}^{0} f_X(a-t)\, dt$$

Let $y = a - t \iff t = a - y$
so $t \in (0, \infty) \iff y \in (a, \infty)$

$$= \int_{\infty}^{a} f_X(y)\,(-1)\, dy$$

$$= \int_{a}^{\infty} f_X(y)\, dy$$

$$= P_X(X \ge a).$$

Since $P_X(X \le a) = P_X(X \ge a)$ and $\underbrace{P_X(X \le a) + P_X(X \ge a)}_{}$ (complements) $= 1$, we have

$$\operatorname{med}(X) = a.$$

(c) Show if $\mathbb{E}X$ exists, $\mathbb{E}X = a$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

Let $x = a+t \iff t = x-a$, so $x \in (-\infty, \infty) \iff t \in (-\infty, \infty)$.

$$= \int_{-\infty}^{\infty} (a+t) \, f_X(a+t) \, dt$$

$$= a \int_{-\infty}^{\infty} f_X(a+t) \, dt + \int_{-\infty}^{\infty} t \, f_X(a+t) \, dt$$

$$= a + \int_{-\infty}^{0} t \, f_X(a+t) \, dt + \int_{0}^{\infty} t \, f_X(a+t) \, dt$$

Let $t = -u \iff dt = (-1)\,du$, u from ∞ to 0

$$= a + \int_{\infty}^{0} (-u) \, f_X(a-u)(-1) \, du + \int_{0}^{\infty} t \, f_X(a+t) \, dt$$

$$= a - \int_{0}^{\infty} u \, f(a-u) \, du + \int_{0}^{\infty} t \, f_X(a+t) \, dt$$

$$= a + \int_{0}^{\infty} t \, [\underbrace{f(a+t) - f(a-t)}_{=0}] \, dt$$

$$= a.$$

(d) Show that $Y \sim f_Y(y) = e^{-y} \mathbb{1}(y \geq 0)$ is not a symmetric pdf.

We find $\text{med}(Y)$ by solving for a in the expression

$$\tfrac{1}{2} = \mathbb{P}_Y(Y \leq a).$$

$$= \int_0^a e^{-y}\, dy$$

$$= -e^{-y} \Big|_0^a$$

$$= 1 - e^{-a}$$

So

$$\mathrm{med}(Y) = \log 2$$

Now $$e^{-(\log 2 - 1)} \mathbb{1}(\underbrace{\log 2 - 1}_{< 0} \geq 0) = 0$$

whereas

$$e^{-(\log 2 + 1)} \mathbb{1}(\log 2 + 1 \geq 0) = 2e^{-1}$$

(e) $$\mathbb{E}\, Y = \int_0^\infty y e^{-y}\, dy = \Gamma(2) = 1 > \log 2 = \mathrm{med}(Y).$$

$\boxed{2.32}$ Let X have mgf $M_X(t)$ and let $S_X(t) = \log M_X(t)$.

(i) Show $\left.\frac{d}{dt} S_X(t)\right|_{t=0} = \mathbb{E}X$.

We have

$$\left.\frac{d}{dt} S_X(t)\right|_{t=0} = \left.\frac{d}{dt} \log M_X(t)\right|_{t=0} = \left.\frac{\frac{d}{dt} M_X(t)}{M_X(t)}\right|_{t=0} = \frac{\mathbb{E}X}{M_X(0)} = \mathbb{E}X,$$

since $M_X(0) = \mathbb{E}e^{0 \cdot X} = \mathbb{E}1 = 1$.

(ii) Show $\left.\frac{d^2}{dt^2} S_X(t)\right|_{t=0} = \operatorname{Var} X$.

We have

$$\left.\frac{d^2}{dt^2} S_X(t)\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{\frac{d}{dt} M_X(t)}{M_X(t)}\right)\right|_{t=0}$$

$$= \left.\frac{\frac{d^2}{dt^2} M_X(t)}{M_X(t)} - \frac{\left[\frac{d}{dt} M_X(t)\right]^2}{\left[M_X(t)\right]^2}\right|_{t=0}$$

$$= \frac{\mathbb{E}X^2}{1} - \frac{\left[\mathbb{E}X\right]^2}{1}$$

$$= \operatorname{Var} X.$$

$\boxed{2.38}$ Let

$$X \sim p_X(x) = \binom{r+x-1}{x} p^r (1-p)^x \mathbb{1}(x \in \{0,1,2,\dots\})$$

for some $0 < p < 1$ and r a positive integer.

(a) Mgf of X is

$$\begin{aligned}
M_X(t) &= \mathbb{E}\, e^{tX} \\
&= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x \\
&= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r [(1-p)e^t]^x \frac{[1-(1-p)e^t]^r}{[1-(1-p)e^t]^r} \\
&= \frac{p^r}{[1-(1-p)e^t]^r} \underbrace{\sum_{x=0}^{\infty} \binom{r+x-1}{x} [(1-p)e^t]^x [1-(1-p)e^t]^r}_{=1 \text{ provided } 1-(1-p)e^t > 0}
\end{aligned}$$

$$\iff t < \log\left(\frac{1}{1-p}\right).$$

So

$$M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r \quad \text{for } t < \log\left(\frac{1}{1-p}\right).$$

(b) Let $Y = 2pX$. Give $\lim_{p \to 0} M_Y(t)$, where M_Y is mgf of Y.

We have

$$M_Y(t) = M_{2pX}(t) = M_X(2pt) = \left(\frac{p}{1-(1-p)e^{2pt}}\right)^r.$$

To take $\lim_{p \to 0} M_Y(t)$, use l'Hôpital's rule:

Theorem 5–11 (L'Hôpital's Rule): Let f and g be differentiable functions on an interval (a, b) with $g'(x) \neq 0$ on (a, b).

(a) Suppose

$$\lim_{x \downarrow a} f(x) = 0, \qquad \lim_{x \downarrow a} g(x) = 0, \qquad \text{and} \qquad \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = L.$$

(b) Suppose

$$\lim_{x \downarrow a} f(x) = \pm\infty, \qquad \lim_{x \downarrow a} g(x) = \pm\infty, \qquad \text{and} \qquad \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = L.$$

← pg 122 of An Introduction to Analysis, 2nd ed by Kirkwood

L'Hôpital's rule gives

$$\lim_{p \to 0} M_Y(t) = \left(\lim_{p \to 0} \frac{\frac{d}{dp}\, p}{\frac{d}{dp}\left[1 - (1-p)e^{2pt}\right]} \right)^r$$

$$= \left(\lim_{p \to 0} \frac{1}{(-1)\left[(1-p)2te^{2pt} - e^{2pt}\right]} \right)^r$$

$$= \left(\frac{1}{1-2t} \right)^r,$$

which is the mgf of the χ^2_{2r} distribution.

$\boxed{2.28}$

(a) For $X \sim f_X$, $f_X(a-\varepsilon) = f_X(a+\varepsilon)$ $\forall \varepsilon > 0$, some a, show $\mathbb{E}(X-a)^3 = 0$.

$$\mathbb{E}(X-a)^3 = \int_{-\infty}^{\infty} (x-a)^3 f_X(x)\, dx$$

$$= \int_{-\infty}^{a} (x-a)^3 f_X(x)\, dx + \int_{a}^{\infty} (x-a)^3 f_X(x)\, dx$$

$$= -\int_{-\infty}^{a} (a-x)^3 f_X(x)\, dx + \int_{a}^{\infty} (x-a)^3 f_X(x)\, dx$$

$u = a - x$
$x = a - u$, $dx = (-1)du$

$v = x - a$
$x = a + v$, $dv = dx$

$$= -\int_{\infty}^{0} u^3 f_X(a-u)(-1)\, du + \int_{0}^{\infty} v^3 f_X(a+v)\, dv$$

$$= -\int_{0}^{\infty} u^3 f_X(a-u)\, du + \int_{0}^{\infty} v^3 f_X(a+v)\, dv$$

$$= \int_{0}^{\infty} u^3 \left[f_X(a+u) - f_X(a-u) \right] du$$

$$= \int_{0}^{\infty} u^3 \cdot 0\, du$$

$$= 0.$$