

STAT 712 hw 4

Transformations of a random variable, mgfs, quantile functions

Do problems 2.1, 2.9, 2.12, 3.13(a), 3.14, 3.25, 3.26 from CB. In addition:

1. Let X be a random variable taking each value in its support $\mathcal{X} = \{x_1, \dots, x_n\}$ with probability $1/n$, and denote by $x_{(1)}, \dots, x_{(n)}$ the values x_1, \dots, x_n when sorted such that $x_{(1)} < x_{(2)} < \dots < x_{(n)}$.
 - (a) Give the quantile function $Q_X(u) = \inf\{x : F_X(x) \geq u\}$, $u \in (0, 1)$, where F_X is the cdf of X .
 - (b) For some $h > 0$, let Y be a random variable with pdf

$$f_Y(y) = \frac{1}{nh} \sum_{i=1}^n \phi\left(\frac{y - x_i}{h}\right),$$

where ϕ is the pdf of the $\text{Normal}(0, 1)$ distribution.

- i. Show that f_Y is a valid pdf.
- ii. Show that the cdf F_Y of Y is given by

$$F_Y(y) = \frac{1}{n} \sum_{i=1}^n \Phi\left(\frac{y - x_i}{h}\right) \quad \text{for all } y \in \mathbb{R},$$

where Φ is the cdf of the $\text{Normal}(0, 1)$ distribution.

- iii. Find $\mathbb{E}Y$.
- iv. Find $\text{Var} Y$.

(c) For $x_i = i$, $i = 1, 2, 3, 4, 5$, make a sketch of F_X with F_Y , for some small h , overlaid.

2. (Optional) Let F_X be a step function with jumps at the points x_1, x_2, \dots and let $Q_X(u) = \inf\{x : F_X(x) \geq u\}$, $u \in (0, 1)$. If $U \sim \text{Uniform}(0, 1)$, show that the rv $X = Q_X(U)$ has cdf F_X .
3. (Optional) 3.24 from CB is time-consuming, but is good for practicing transformations. I highly recommend working out all parts of this question.