STAT 712 hw 5

Joint and marginal distributions, conditional distributions, independence

Do problems 4.1, 4.9, 4.10, 4.11, 4.15 from CB. In addition:

1. A frog will hop across a sidewalk, beginning the dirt on one side and ending in the dirt on the other side. Let X be the number of times the frog lands on the sidewalk while hopping across. Derive the probability mass function of X assuming that the frog's hopping distances are independent and have the exponential distribution with mean $1/\lambda$ and that the sidewalk has width t.

2. Let (X, Y) be a pair of random variables with joint pdf given by
$$f(x,y) = \frac{x}{\theta}e^{-x/\theta}\mathbf{1}(0 < y < 1/x, x > 0)$$
for some $\theta > 0$.
   (a) Find $P(1 \leq X \leq 2, Y \leq 1)$.
   (b) Find the marginal pdf f_X of X.
   (c) Find $\mathbb{E}X$.
   (d) Find the marginal pdf f_Y of Y and draw a picture of it when $\theta = 1$ (you may use software). *Hint: You will have to do integration by parts.*
   (e) Give the conditional pdf $f(x|y)$ of $X|Y = y$ for $y = 1$ when $\theta = 1$.
   (f) Give the conditional pdf $f(y|x)$ of $Y|X = x$ for $x > 0$.

3. Let (Z_1, Z_2) be a pair of rvs with the standard bivariate Normal distribution with correlation ρ, so that their joint pdf is given by
$$f(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right] \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$
   (a) Show that Z_1 and Z_2 are independent if $\rho = 0$.
   (b) Show that the marginal pdf of Z_1 is the $\text{Normal}(0,1)$ distribution.
   (c) Show that $Z_2|Z_1 = z_1 \sim \text{Normal}(\rho z_1, 1-\rho^2)$.

4. Let X have pdf f_X and for some $\tau \in (0,1)$ define the *quantile check function* as
$$\rho_\tau(z) = z(\tau - \mathbf{1}(z < 0)) = \begin{cases} z\tau, & z \geq 0 \\ -z(1-\tau), & z < 0. \end{cases}$$
   (a) Show that the τ-quantile q_τ of X is equal to the value of a which minimizes $\mathbb{E}\rho_\tau(X - a)$. *Hint: Set up the integral and differentiate it with respect to a using the rule of Leibniz*
$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,t)dt = g(x, b(x))\frac{d}{dx}b(x) - g(x, a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{d}{dx}g(x,t)dt.$$
   Then show that the derivative is equal to zero when $a = q_\tau$.
   (b) Argue that the median of X is the value of a which minimizes $\mathbb{E}|X - a|$.

5. (Optional) Additional problems from CB: 4.4, 4.5, 4.17, 4.18