STAT 712 hw 6

Bivariate transformations, sums of independent rvs

Do problems 4.19, 4.21, 4.26, from CB. *Hint for 4.26: Get the conditional cdf of $Z|W = 0$ and the conditional cdf of $Z|W = 1$. This is fully describes the "joint distribution" of (Z, W).* In addition:

1. Let (Z_1, Z_2) be a pair of rvs with the standard bivariate Normal distribution with correlation ρ, so that their joint pdf is given by

$$f(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right] \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

   (a) Find the joint density of $U_1 = Z_1 - Z_2$ and $U_2 = Z_1 + Z_2$.

   (b) Check whether U_1 and U_2 are independent.

2. Let $Z \sim \text{Normal}(0, 1)$ and $W \sim \chi^2_\nu$, with $\nu > 0$, be independent random variables.

   (a) Show that

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu,$$

   where t_ν represents the t-distribution with ν degrees of freedom, which has pdf given by

$$f_T(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{for } t \in \mathbb{R}.$$

   Hint: First find the joint density of (T, U), where $U = W$.

   (b) Find the mean of a random variable having the t_ν distribution.

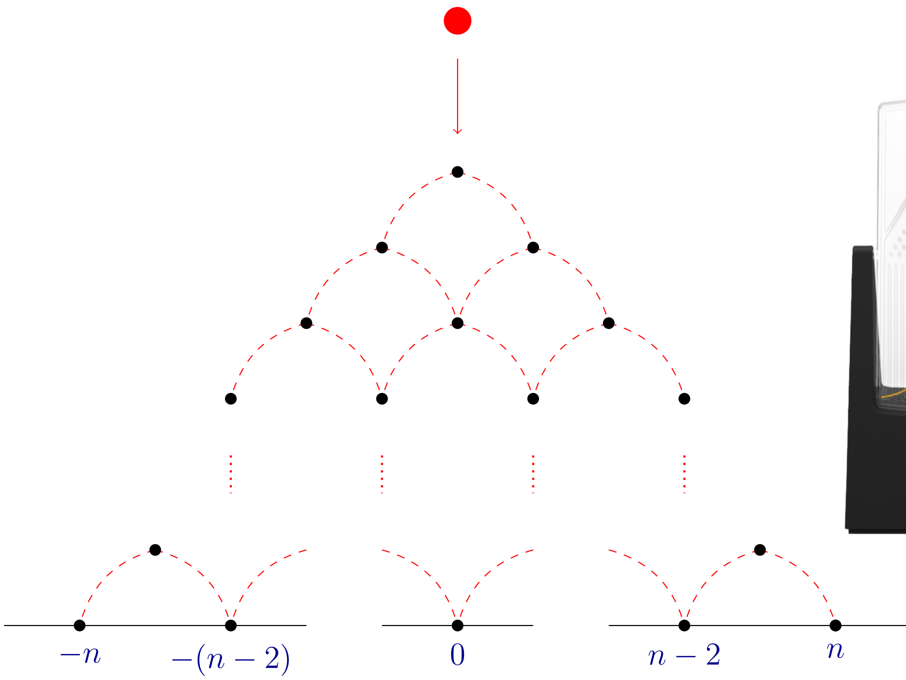
   (c) Show that $\mathbb{E}W^{-1} = 1/(\nu - 2)$, assuming $\nu > 2$.

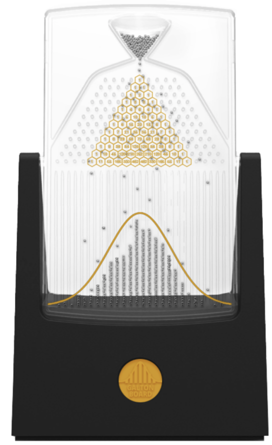
   (d) Find the variance of a random variable having the t_ν distribution.

3. Let $X \sim f_X$ and $Y \sim f_Y$ be independent rvs. Show that the pdf of $V = X + Y$ is given by the convolution of f_X and f_Y; that is

$$f_V(v) = (f_X * f_Y)(v) = \int_{-\infty}^{\infty} f_X(v-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(y)f_Y(v-y)dy \quad \text{for all} \quad v \in \mathbb{R}.$$

4. (Optional) Suppose a ball is pulled by gravity through a triangular lattice of pins, such that at each pin it must go to the left or the right, as figured below. The first pin is positioned at 0, the two pins in the next row at -1 and 1, and the three pins in the next row at -2, 0, and 2, and so on.





Let Y be the position of the pin the ball touches on its nth hit.

(a) Give the pmf of Y in the case that n is even and in the case that n is odd.

(b) Find the pmf p_X such that $Y \stackrel{d}{=} X_1 + \cdots + X_n$ when $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} p_X$.

(c) Give the mgf M_X corresponding to p_X from the previous part. Write M_X as an infinite series.

(d) Show that $n^{-1/2}Y$ converges in distribution to the $\text{Normal}(0, 1)$ distribution.

5. (Optional) Let $W_1 \sim \chi^2_{\nu_1}$ and $W_2 \sim \chi^2_{\nu_2}$, with $\nu_1 > 0$ and $\nu_2 > 0$, be independent random variables. Show that

$$R = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1,\nu_2},$$

where F_{ν_1,ν_2} represents the F-distribution with numerator degrees of freedom ν_1 and denominator degrees of freedom ν_2, which has pdf given by

$$f_R(r; \nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2}\left(1 + \frac{\nu_1}{\nu_2}r\right)^{-(\nu_1+\nu_2)/2} \quad \text{for } r > 0.$$

Hint: First find the joint density of (R, U), where $U = W_2$.