STAT 712 hw 6

Bivariate transformations, sums of independent rvs

Do problems 4.19, 4.21, 4.26, from CB. *Hint for 4.26: Get the conditional cdf of* $Z|W = 0$ *and the conditional cdf of* $Z|W = 1$*. This is fully describes the "joint distribution" of* (Z, W)*.* In addition:

1. Let (Z_1, Z_2) be a pair of rvs with the standard bivariate Normal distribution with correlation ρ, so that their joint pdf is given by

$$f(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right] \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

(a) Find the joint density of $U_1 = Z_1 - Z_2$ and $U_2 = Z_1 + Z_2$.

Firstly, note that (U_1, U_2) has support on $\mathbb{R} \times \mathbb{R}$. Now, we have

$$\begin{array}{l} u_1 = z_1 + z_2 =: g_1(z_1, z_2) \\ u_2 = z_1 - z_2 =: g_2(z_1, z_2) \end{array} \iff \begin{array}{l} z_1 = (u_1 + u_2)/2 =: g_1^{-1}(u_1, u_2) \\ z_2 = (u_1 - u_2)/2 =: g_2^{-1}(u_1, u_2) \end{array}$$

with Jacobian

$$J(x, y) = \begin{vmatrix} \frac{d}{du_1}(u_1 + u_2)/2 & \frac{d}{du_2}(u_1 + u_2)/2 \\ \frac{d}{du_1}(u_1 - u_2)/2 & \frac{d}{du_2}(u_1 - u_2)/2 \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2.$$

The transformation method gives

$$\begin{aligned} f(u_1, u_2) &= \frac{1}{2\pi}\frac{1}{1-\rho^2} \\ &\times \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{u_1+u_2}{2}\right)^2 - 2\rho\left(\frac{u_1+u_2}{2}\right)\left(\frac{u_1-u_2}{2}\right) + \left(\frac{u_1-u_2}{2}\right)^2\right)\right] |-\frac{1}{2}| \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2+2\rho}} \exp\left[-\frac{u_1^2}{2(2+2\rho)}\right] \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2-2\rho}} \exp\left[-\frac{u_1^2}{2(2-2\rho)}\right], \end{aligned}$$

where the simplification takes a few steps.

(b) Check whether U_1 and U_2 are independent.

We can see from the joint density that U_1 and U_2 are independent, because of the factorization theorem; moreover, we can identify the distributions of U_1 and U_2 as

$$\begin{aligned} U_1 &\sim \text{Normal}(0, 2 + 2\rho) \\ U_2 &\sim \text{Normal}(0, 2 - 2\rho). \end{aligned}$$

2. Let $Z \sim \text{Normal}(0, 1)$ and $W \sim \chi^2_\nu$, with $\nu > 0$, be independent random variables.

(a) Show that

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu,$$

where t_ν represents the t-distribution with ν degrees of freedom, which has pdf given by

$$f_T(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \qquad \text{for } t \in \mathbb{R}.$$

Hint: First find the joint density of (T, U), where $U = W$.

(b) Find the mean of a random variable having the t_ν distribution.

We have $\mathbb{E}\frac{Z}{\sqrt{W/\nu}} = \sqrt{\nu}\mathbb{E}X\mathbb{E}W^{-1/2} = 0$, where we have used that fact that Z and W are independent and $\mathbb{E}Z = 0$.

(c) Show that $\mathbb{E}W^{-1} = 1/(\nu - 2)$, assuming $\nu > 2$.

We have

$$\begin{aligned}
\mathbb{E}W^{-1} &= \int_0^\infty \frac{1}{w} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} w^{\nu/2-1} e^{-w/2} dw \\
&= \frac{\Gamma(\nu/2-1)2^{\nu/2-1}}{\Gamma(\nu/2)2^{\nu/2}} \underbrace{\int_0^\infty \frac{1}{\Gamma(\nu/2-1)2^{\nu/2-1}} w^{(\nu/2-1)-1} e^{-w/2} dw}_{=1} \\
&= \frac{\Gamma(\nu/2-1)}{(\nu/2-1)\Gamma(\nu/2-1)2} \qquad \text{(property of Gamma function)} \\
&= 1/(\nu-2).
\end{aligned}$$

(d) Find the variance of a random variable having the t_ν distribution.

Since $\mathbb{E}T = 0$, we have $\operatorname{Var} T = \mathbb{E}T^2 = \nu\mathbb{E}Z^2\mathbb{E}W^{-1} = \nu/(\nu-2)$, since $\mathbb{E}Z^2 = 1$.

3. Let $X \sim f_X$ and $Y \sim f_Y$ be independent rvs. Show that the pdf of $V = X + Y$ is given by the convolution of f_X and f_Y; that is

$$f_V(v) = (f_X * f_Y)(v) = \int_{-\infty}^{\infty} f_X(v-y) f_Y(y) dy = \int_{-\infty}^{\infty} f_X(y) f_Y(v-y) dy \quad \text{for all} \quad v \in \mathbb{R}.$$

Since X and Y are independent, the joint density of (X, Y) is given by $f(x, y) = f_X(x)f_Y(y)$. Now, let $V = X + Y$ and $U = Y$. Then we have

$$\begin{array}{l} v = x + y =: g_1(x, y) \\ u = y =: g_2(x, y) \end{array} \quad \Longleftrightarrow \quad \begin{array}{l} x = v - u =: g_1^{-1}(u, v) \\ y = u =: g_2^{-1}(u, v) \end{array}$$

with Jacobian

$$J(x, y) = \begin{vmatrix} \frac{d}{du}(v-u) & \frac{d}{dv}(v-u) \\ \frac{d}{du}u & \frac{d}{dv}u \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

So the joint density of (U, V) is given by

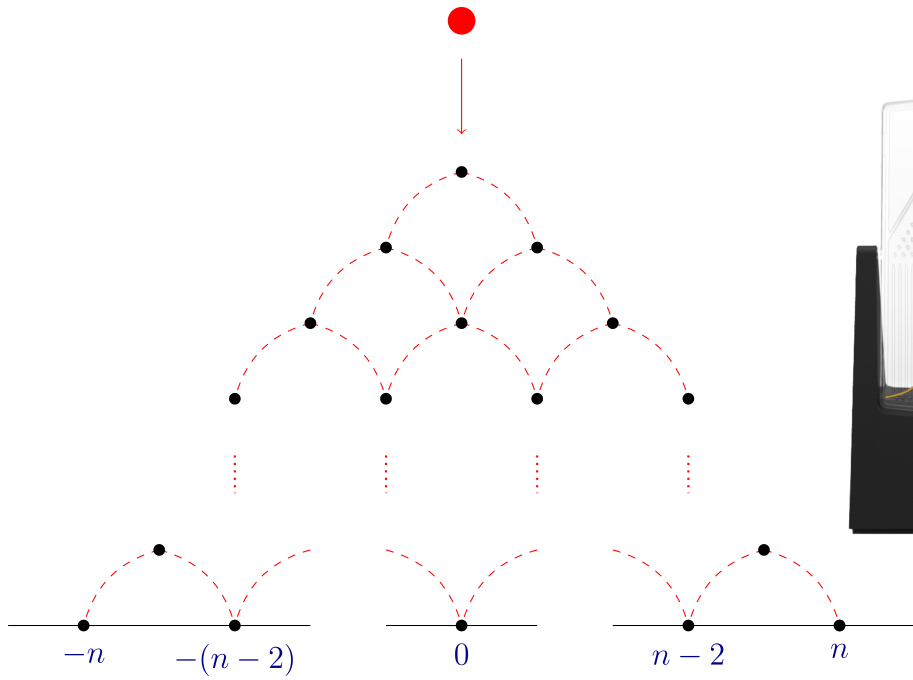
$$f(u, v) = f_X(v-u)f_Y(u)|-1| = f_X(v-u)f_Y(u).$$

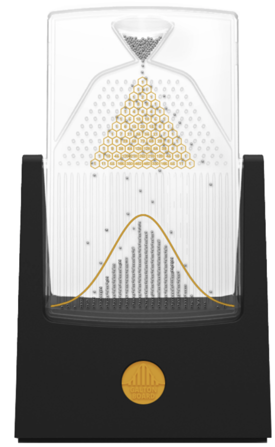
From here we obtain the marginal density of V as

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v-u)f_Y(u)du,$$

which gives the first equality (we can write y instead of u if we wish). To get the second inequality, do the same for the bivariate transformation $V = X + Y$ and $U = X$.

4. (Optional) Suppose a ball is pulled by gravity through a triangular lattice of pins, such that at each pin it must go to the left or the right, as figured below. The first pin is positioned at 0, the two pins in the next row at -1 and 1, and the three pins in the next row at -2, 0, and 2, and so on.





Let Y be the position of the pin the ball touches on its nth hit.

(a) Give the pmf of Y in the case that n is even and in the case that n is odd.

> We have
>
> $$p_Y(y) = \begin{cases} \binom{n}{(n-y)/2}(1/2)^n \cdot \mathbf{1}(y = 0, \pm 2, \pm 4, \dots, \pm n), & \text{if } n \text{ is even} \\ \binom{n}{(n-y)/2}(1/2)^n \cdot \mathbf{1}(y = \pm 1, \pm 3, \dots, \pm n), & \text{if } n \text{ is odd.} \end{cases}$$

(b) Find the pmf p_X such that $Y \overset{d}{=} X_1 + \cdots + X_n$ when $X_1, \dots, X_n \overset{\text{ind}}{\sim} p_X$.

> We find that if $X_1, \dots, X_n \overset{\text{ind}}{\sim} p_X(x) = (1/2)\mathbf{1}(x \in \{-1, 1\})$, then $Y \overset{d}{=} X_1 + \cdots + X_n$. The random variables $X_1, \dots, X_n$ are called Rademacher random variables.

(c) Give the mgf M_X corresponding to p_X from the previous part. Write M_X as an infinite series.

With $p_X(x) = (1/2)\mathbf{1}(x \in \{-1, 1\})$, we find

$$\begin{aligned} M_X(t) &= (1/2)[e^t + e^{-t}] \\ &= (1/2)[1 + t + t^2/2! + t^3/3! + t^4/4! + \dots \\ &\qquad\qquad + 1 - t + t^2/2! - t^3/3! + t^4/4! - \dots] \\ &= 1 + t^2/2! + t^4/4! + \dots \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}. \end{aligned}$$

(d) Show that $n^{-1/2}Y$ converges in distribution to the $\text{Normal}(0, 1)$ distribution.

We have

$$\begin{aligned} M_{Y/\sqrt{n}}(t) &= M_{X_1+\dots+X_n}(t/\sqrt{n}) \\ &= [M_X(t/\sqrt{n})]^n \\ &= \left[\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right]^n \\ &= [1 + (t/\sqrt{n})^2/2! + (t/\sqrt{n})^4/4! + \dots]^n \\ &= [1 + (t^2/2)/n + (t^4/4!)/n^2 + \dots]^n \\ &\to e^{t^2/2} \end{aligned}$$

as $n \to \infty$, where we recognize $e^{t^2/2}$ as the mgf of the $\text{Normal}(0, 1)$ distribution.

5. (Optional) Let $W_1 \sim \chi^2_{\nu_1}$ and $W_2 \sim \chi^2_{\nu_2}$, with $\nu_1 > 0$ and $\nu_2 > 0$, be independent random variables. Show that

$$R = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1,\nu_2},$$

where F_{ν_1,ν_2} represents the F-distribution with numerator degrees of freedom ν_1 and denominator degrees of freedom ν_2, which has pdf given by

$$f_R(r; \nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2} \left(1 + \frac{\nu_1}{\nu_2} r\right)^{-(\nu_1+\nu_2)/2} \quad \text{for } r > 0.$$

Hint: First find the joint density of (R, U), where $U = W_2$.

First, the joint pdf of (W_1, W_2) is given by

$$f(w_1, w_2) = \frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}} w_1^{\frac{\nu_1}{2}-1} e^{-\frac{w_1}{2}} \frac{1}{\Gamma(\frac{\nu_2}{2})2^{\frac{\nu_2}{2}}} w_2^{\frac{\nu_2}{2}-1} e^{-\frac{w_2}{2}} \mathbf{1}(w_1 > 0, w_2 > 0).$$

The support of (R, U) is $(0, \infty) \times (0, \infty)$. We have

$$\begin{array}{ll} r = (w_1/\nu_1)/(w_2/\nu_2) =: g_1(w_1, w_2) \\ u = w_2 =: g_2(w_1, w_2) \end{array} \iff \begin{array}{ll} w_1 = ru(\nu_1/\nu_2) =: g_1^{-1}(r, u) \\ w_2 = u =: g_2^{-1}(r, u) \end{array}$$

with Jacobian

$$J(x, y) = \begin{vmatrix} \frac{d}{dr} ru(\nu_1/\nu_2) & \frac{d}{du} ru(\nu_1/\nu_2) \\ \frac{d}{dr} u & \frac{d}{du} u \end{vmatrix} = \begin{vmatrix} u(\nu_1/\nu_2) & r(\nu_1/\nu_2) \\ 0 & 1 \end{vmatrix} = u(\nu_1/\nu_2).$$

Now we have

$$\begin{aligned} f(r, u) &= \frac{1}{\Gamma(\frac{\nu_1}{2}) 2^{\frac{\nu_1}{2}}} \left(ru\frac{\nu_1}{\nu_2} \right)^{\frac{\nu_1}{2}-1} e^{-\frac{1}{2} ru \frac{\nu_1}{\nu_2}} \frac{1}{\Gamma(\frac{\nu_2}{2}) 2^{\frac{\nu_2}{2}}} u^{\frac{\nu_2}{2}-1} e^{-\frac{u}{2}} |u\frac{\nu_1}{\nu_2}| \mathbf{1}(r > 0, u > 0) \\ &= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \frac{r^{\frac{\nu_1}{\nu_2}-1}}{2^{\frac{\nu_1+\nu_2}{2}}} u^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{u}{2}\left(1 + r\frac{\nu_1}{\nu_2}\right)} \mathbf{1}(r > 0, u > 0) \end{aligned}$$

Now we integrate the above over u to obtain the marginal pdf of R. For any $r > 0$, we have

$$\begin{aligned} f_R(r) &= \int_0^\infty \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \frac{r^{\frac{\nu_1}{\nu_2}-1}}{2^{\frac{\nu_1+\nu_2}{2}}} u^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{u}{2}\left(1 + r\frac{\nu_1}{\nu_2}\right)} du \\ &= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{r^{\frac{\nu_1-2}{2}}}{\left(1 + r\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}}, \end{aligned}$$

which is the pdf of the F_{ν_1,ν_2} distribution.

Problems 4.19, 4.21, and 4.26 from CB.

$\boxed{4.19}$

(a) Let $X_1, X_2 \overset{iid}{\sim} N(0,1)$.

Then $W = \frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)$, since

$$
\begin{aligned}
M_W(t) &= M_{\frac{X_1 - X_2}{\sqrt{2}}}(t) \\
&= \mathbb{E}\, e^{\frac{t(X_1 - X_2)}{\sqrt{2}}} \\
&= \mathbb{E}\, e^{(t/\sqrt{2})X_1} e^{(-t/\sqrt{2})X_2} \\
&= M_{X_1}(t/\sqrt{2})\, M_{X_2}(-t/\sqrt{2}) \\
&= e^{(t/\sqrt{2})^2/2} e^{(-t/\sqrt{2})^2/2} \\
&= e^{t^2/2}.
\end{aligned}
$$

Now $\frac{(X_1 - X_2)^2}{2} = W^2 \sim \chi^2_1$, as we have shown before.

(b) Let $X_1 \sim \text{Gamma}(\alpha_1, 1)$, $X_2 \sim \text{Gamma}(\alpha_2, 1)$ be indep rvs.

We have

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-x_1} \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-x_2} \mathbb{1}(x_1 > 0, x_2 > 0).$$

Now let

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad \text{and} \quad Y_2 = X_1 + X_2.$$

Then $(Y_1, Y_2) \in (0,1) \times (0, \infty)$.

We have

$$Y_1 = g_1(X_1, X_2) = \frac{X_1}{X_1+X_2} \qquad\qquad X_1 = Y_1 Y_2 = g_1^{-1}(Y_1,Y_2)$$
$$\iff$$
$$Y_2 = g_2(X_1,X_2) = X_1 + X_2 \qquad\qquad X_2 = Y_2 - Y_1Y_2 = g_2^{-1}(Y_1,Y_2)$$

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial}{\partial y_1} y_1 y_2 & \frac{\partial}{\partial y_1}(y_2 - y_1y_2) \\ \frac{\partial}{\partial y_2} y_1y_2 & \frac{\partial}{\partial y_2}(y_2 - y_1y_2) \end{vmatrix}$$

$$= \begin{vmatrix} y_2 & -y_2 \\ y_1 & 1-y_1 \end{vmatrix}$$

$$= (1-y_1)y_2 + y_1y_2$$

$$= y_2 .$$

Now the bivariate transformation method gives

$$f(y_1,y_2) = \frac{1}{\Gamma(\alpha_1)}(y_1y_2)^{\alpha_1-1} e^{-y_1y_2} \frac{1}{\Gamma(\alpha_2)}(y_2-y_1y_2)^{\alpha_2-1} e^{-(y_2-y_1y_2)} \, y_2 \, \mathbb{1}(0<y_1<1,\ y_2>0)$$

$$= \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\mathbb{1}(0<y_1<1)}_{\text{Beta}(\alpha_1,\alpha_2)} \ \underbrace{\frac{y_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} e^{-y_2}\mathbb{1}(y_2>0)}_{\text{Gamma}(\alpha_1+\alpha_2)}$$

So the marginal of Y_1 is

$$f_{Y_1}(y_1) = \int_0^\infty \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\mathbb{1}(0<y_1<1) \quad \frac{y_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} e^{-y_2} \, dy_2$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\mathbb{1}(0<y_1<1).$$

So

$$\frac{X_1}{X_1+X_2} \sim \text{Beta}(\alpha_1, \alpha_2).$$

Likewise we can show that

$$\frac{X_2}{X_1+X_2} \sim \text{Beta}(\alpha_2, \alpha_1).$$

$\boxed{4.21}$ Let $R^2 \sim \chi^2_2$ and $\Theta \sim \text{Unif}(0, 2\pi)$ be indep. rvs.

Find the joint density of $(X, Y) = (R\cos\Theta, R\sin\Theta)$

Let $W = R^2$. Then we wish to find the joint density of

$$(X, Y) = (\sqrt{W}\cdot\cos\Theta, \sqrt{W}\cdot\sin\Theta).$$

Recall $\chi^2_2 = \text{Gamma}(\frac{2}{2}, 2) = \text{Gamma}(1, 2)$, so

$$f(w, \theta) = \frac{1}{2\pi}\frac{1}{\Gamma(1)2^1} w^{1-1} e^{-w/2} \quad \mathbb{1}(0<\theta<2\pi, w>0)$$

$$= \frac{1}{2\pi}\frac{1}{2} e^{-w/2} \mathbb{1}(0<\theta<2\pi, w>0)$$

Now

$$x = \sqrt{w}\cos\theta = g_1(\theta, w) \qquad \Longleftrightarrow \qquad w = x^2 + y^2 = g_1^{-1}(x,y)$$

$$y = \sqrt{w}\sin\theta = g_2(\theta, w) \qquad\qquad \theta = \underbrace{\tan^{-1}(y/x) + \pi\,\mathbb{1}(x<0) + 2\pi\,\mathbb{1}(x<0,\, y<0)}_{\text{weirdness because } \tan^{-1}:\,\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}$$

As long as $x \neq 0$ and $y \neq 0$, we have

$$J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x}(x^2+y^2) & \frac{\partial}{\partial x}\tan^{-1}(y/x) \\ \frac{\partial}{\partial y}(x^2+y^2) & \frac{\partial}{\partial y}\tan^{-1}(y/x) \end{vmatrix}$$

$$= \begin{vmatrix} 2x & \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) \\ 2y & \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{1}{x} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -\frac{y}{x^2+y^2} \\ 2y & \frac{1}{x+y^2/x} \end{vmatrix}$$

$$= \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}$$

$$= 2$$

So we have

$$f(x_1,x_2) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$\boxed{4.26}$ Let $X \sim \text{Exponential}(\lambda)$, $Y \sim \text{Exponential}(\mu)$.

Set $Z = \min\{X, Y\}$ and $W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y. \end{cases}$

Figure out joint distribution of Z and W.

W is a Bernoulli with success prob $P(X < Y)$. We have

$$P(X < Y) = \int_0^\infty \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} \, dy \, dx$$

$$= \int_0^\infty \frac{1}{\lambda} e^{-x/\lambda} \left[-e^{-y/\mu} \right] \Big|_x^\infty dx$$

$$= \int_0^\infty \frac{1}{\lambda} e^{-x/\lambda} e^{-x/\mu} \, dx$$

$$= \frac{1}{\lambda} \int_0^\infty e^{-x\left(\frac{\lambda+\mu}{\lambda\mu}\right)} dx$$

$$= \frac{1}{\lambda} \left[-\left(\frac{\lambda\mu}{\lambda+\mu}\right) e^{-x\left(\frac{\lambda+\mu}{\lambda\mu}\right)} \right] \Big|_0^\infty$$

$$= \frac{\mu}{\lambda+\mu}.$$

Now consider

$$P(X \le x \mid X < Y) = 1 - P(X > x \mid X < Y)$$

$$= 1 - \frac{P(x < X < Y)}{P(X < Y)}.$$

For any $x > 0$, we have

$$P(x < X < Y) = \int_x^\infty \int_{t_1}^\infty \frac{1}{\lambda} e^{-\frac{t_1}{\lambda}} \frac{1}{\mu} e^{-\frac{t_2}{\mu}} \, dt_2 \, dt_1$$

$$= \int_x^\infty \frac{1}{\lambda} e^{-t_1/\lambda} \left[-e^{-t_2/\mu} \right] \Big|_{t_1}^\infty dt_1$$

$$= \int_x^\infty \frac{1}{\lambda} e^{-t_1/\lambda} \left[e^{-t_1/\mu} \right] dt_1$$

$$= \frac{1}{\lambda} \int_x^\infty e^{-t_1 \left(\frac{\lambda+\mu}{\lambda\mu} \right)} dt_1$$

$$= \frac{1}{\lambda} \left[-\left(\frac{\lambda\mu}{\lambda+\mu} \right) e^{-t_1 \left(\frac{\lambda+\mu}{\lambda\mu} \right)} \right] \Big|_x^\infty$$

$$= \frac{\mu}{\lambda+\mu} e^{-x \left(\frac{\lambda+\mu}{\lambda\mu} \right)} .$$

So

$$P(X \le x \mid X < Y) = 1 - \frac{\frac{\mu}{\lambda+\mu} e^{-x \left(\frac{\lambda+\mu}{\lambda\mu} \right)}}{\frac{\mu}{\lambda+\mu}}$$

$$= 1 - e^{-x \left(\frac{\lambda+\mu}{\lambda\mu} \right)} .$$

This is the conditional cdf of Z given that $W=1$ $(Z=X)$, so

$$f_{Z|W=1}(z) = \left(\frac{\lambda+\mu}{\lambda\mu}\right) e^{-z\left(\frac{\lambda+\mu}{\lambda\mu}\right)} \mathbb{1}(z>0).$$

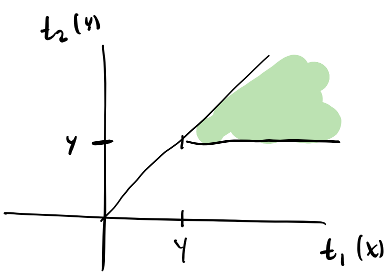
Now consider

$$P(Y \le y \mid Y < X) = 1 - P(Y > y \mid Y < X)$$

$$= 1 - \frac{P(y < Y < X)}{P(Y<X)}.$$

For any $y > 0$ we have

$$P(y < Y < X) = \int_y^\infty \int_{t_2}^\infty \frac{1}{\lambda} e^{-t_1/\lambda} \frac{1}{\mu} e^{-\frac{t_2}{\mu}} \, dt_1 \, dt_2$$



$$= \frac{1}{\mu} \int_y^\infty e^{-t_2/\mu} \left[-e^{-t_1/\lambda}\right]\Big|_{t_2}^\infty \, dt_2$$

$$= \frac{1}{\mu} \int_y^\infty e^{-t_2/\mu} \, e^{-t_2/\lambda} \, dt_2$$

$$= \frac{1}{\mu} \int_y^\infty e^{-t_2\left(\frac{\lambda+\mu}{\lambda\mu}\right)} \, dt_2$$

$$= \frac{1}{\mu} \left[-\left(\frac{\lambda\mu}{\lambda+\mu}\right) e^{-t_2\left(\frac{\lambda+\mu}{\lambda\mu}\right)}\right]\Bigg|_y^\infty$$

$$= \frac{\lambda}{\lambda+\mu} e^{-y\left(\frac{\lambda+\mu}{\lambda\mu}\right)}$$

So we have

$$P(Y \le y \mid Y < X) = 1 - \frac{\frac{\lambda}{\lambda+\mu} e^{-y\left(\frac{\lambda+\mu}{\lambda\mu}\right)}}{1 - \frac{\mu}{\lambda+\mu}}$$

$$= 1 - e^{-y\left(\frac{\lambda+\mu}{\lambda\mu}\right)}.$$

This is the conditional cdf of Z given that $W=0$ $(Z=Y)$, so

$$f_{Z|W=0}(z) = \left(\frac{\lambda+\mu}{\lambda\mu}\right) e^{-z\left(\frac{\lambda+\mu}{\lambda\mu}\right)} \mathbb{1}(z>0).$$

Since the conditional distribution of $Z|W=1$ is the same as that of $Z|W=0$, Z is independent of W.

So we can describe the joint distribution of the rv pair (Z,W) as

$$Z \sim f_Z(z) = \left(\frac{\lambda+\mu}{\lambda\mu}\right) e^{-z\left(\frac{\lambda+\mu}{\lambda\mu}\right)} \mathbb{1}(z>0)$$

$$W \sim \text{Bernoulli}\left(\frac{\mu}{\lambda+\mu}\right)$$

with $Z \perp\!\!\!\perp W$.