STAT 712 hw 7

Covariance, hierarchical models, inequalities

Do problems 4.32, 4.42, 4.43, 4.54, 4.58, 4.63 from CB.

1. For $Z_1, \dots, Z_p \sim \text{Normal}(0, 1)$, not necessarily independent, prove the maximal inequality

$$\mathbb{E} \max_{1 \le j \le p} Z_j \le \sqrt{2 \log p}.$$

Use these steps:

(a) Show that for all $t \in \mathbb{R}$ we have $\exp(t \cdot \mathbb{E} \max_{1 \le j \le p} Z_j) \le p e^{t^2/2}$. *Hint: Begin with Jensen's.*

(b) Find the value of t yielding the best possible upper bound on $\mathbb{E} \max_{1 \le j \le p} Z_j$.

2. Let $X_1 \sim \text{Normal}(m_1, \kappa^{-1})$ and $X_2 \sim \text{Normal}(m_2, \kappa^{-1})$ be independent rvs.

(a) Let the rv pair (R, Θ) be defined by $X_1 = R \cos \Theta$ and $X_2 = R \sin \Theta$, where $R > 0$ and $\Theta \in [-\pi, \pi)$. In addition, represent m_1 and m_2 as $m_1 = s \cdot \cos \mu$ and $m_2 = s \cdot \sin \mu$ for some $s > 0$ and $\mu \in [-\pi, \pi)$. Give the joint pdf of (R, Θ).

(b) Note that if $s = 1$ the point (m_1, m_2) lies on the unit circle and if $R = 1$ the point (X_1, X_2) lies on the unit circle. Show that, under $s = 1$, the conditional density of Θ given $R = 1$ is given by

$$f(\theta | R = 1) = \frac{e^{\kappa \cos(\theta - \mu)}}{\int_{-\pi}^{\pi} e^{\kappa \cos(\theta' - \mu)} d\theta'} \mathbf{1}(-\pi \le \theta < \pi).$$

This is the pdf of the von Mises distribution, which is often used for modeling directional data.

3. (Optional) Additional problems from CB: 4.33, 4.40.