STAT 712 hw 8

Random samples, Normal-population pivot quantities

Do problems 5.8(a), 5.10, 5.12, 5.13, 5.16, 5.17(b,c,d) from CB. In addition:

1. Let $W \sim \chi^2_\nu(\phi)$, so that W has the non-central chi-squared distribution with degrees of freedom $\nu > 0$ and non-centrality parameter $\phi > 0$. The pdf of W is given by

$$f_W(w;\nu,\phi) = \sum_{k=0}^{\infty} \frac{e^{-\phi/2}(\phi/2)^k}{k!} \frac{1}{\Gamma(\frac{\nu+2k}{2})2^{\frac{\nu+2k}{2}}} w^{\frac{\nu+2k}{2}-1} e^{-w/2} \mathbf{1}(w > 0).$$

(a) Find the mgf of W.

We have

$$\begin{aligned}
\mathbb{E}e^{tW} &= \int_0^\infty e^{tw} \sum_{k=0}^{\infty} \frac{e^{-\phi/2}(\phi/2)^k}{k!} \frac{1}{\Gamma(\frac{\nu+2k}{2})2^{\frac{\nu+2k}{2}}} w^{\frac{\nu+2k}{2}-1} e^{-w/2} dw \\
&= \sum_{k=0}^{\infty} \frac{(\frac{1}{2}-t)^{-\frac{\nu+2k}{2}}}{2^{\frac{\nu+2k}{2}}} \frac{e^{-\phi/2}(\phi/2)^k}{k!} \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{\nu+2k}{2})} \frac{1}{(\frac{1}{2}-t)^{-\frac{\nu+2k}{2}}} w^{(\frac{\nu+2k}{2})-1} e^{-w/(\frac{1}{2}-t)^{-1}} dw}_{=1 \text{ provided } t<1/2} \\
&= \sum_{k=0}^{\infty} (1-2t)^{-\frac{\nu+2k}{2}} \cdot \frac{e^{-\phi/2}(\phi/2)^k}{k!} \\
&= (1-2t)^{-\frac{\nu}{2}} \cdot \frac{e^{-\phi/2}}{e^{-(\frac{\phi/2}{1-2t})}} \underbrace{\sum_{k=0}^{\infty} \frac{e^{-(\frac{\phi/2}{1-2t})}(\frac{\phi/2}{1-2t})^k}{k!}}_{=1} \\
&= (1-2t)^{-\frac{\nu}{2}} e^{\phi t/(1-2t)}.
\end{aligned}$$

We note that we need $t < 1/2$, so we have

$$M_W(t) = (1-2t)^{-\frac{\nu}{2}} e^{\phi t/(1-2t)} \quad \text{for } t < 1/2.$$

(b) Let $W_1, \dots, W_q$ be independent rvs such that $W_j \sim \chi^2_{\nu_j}(\phi_j)$ for $j = 1, \dots, q$. Find the mgf of $V = W_1 + \cdots + W_q$ and identify the distribution to which it belongs.

The mgf of W_j is given by

$$M_{W_j}(t) = (1-2t)^{-\frac{\nu_j}{2}} e^{\phi_j t/(1-2t)} \quad \text{for } t < 1/2$$

for $j = 1, \dots, q$. Since $W_1, \dots, W_q$ are independent, we have

$$M_V(t) = \prod_{j=1}^{q} (1-2t)^{-\frac{\nu_j}{2}} e^{\phi_j t/(1-2t)}$$

$$= (1-2t)^{-\frac{\sum_{j=1}^{q} \nu_j}{2}} e^{(\sum_{j=1}^{q} \phi_j) t/(1-2t)},$$

which we recognize as the mgf of the $\chi^2_{\sum_{j=1}^{q} \nu_j}(\sum_{j=1}^{q} \phi_j)$ distribution.

2. (Optional) Additional problems from CB: 5.8 (b)(c), 5.11

Problems 5.8, 5.10, 5.12, 5.13, 5.16, 5.17 from CB.

$\boxed{5.8}$

(a) We have

$$\frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left((x_i - \bar{x}_n) - (x_j - \bar{x}_n) \right)^2$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ (x_i - \bar{x}_n)^2 + (x_j - \bar{x}_n)^2 - 2(x_i - \bar{x}_n)(x_j - \bar{x}_n) \right\}$$

$$= \frac{1}{2n(n-1)} \left\{ \sum_{i=1}^{n} n(x_i - \bar{x}_n)^2 + \sum_{j=1}^{n} n(x_j - \bar{x}_n)^2 - \right.$$

$$\left. - \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - \bar{x}_n)(x_j - \bar{x}_n) \right\}$$

$$= \frac{1}{2(n-1)} \left\{ \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 + \sum_{j=1}^{n} (x_j - \bar{x}_n)^2 \right\}$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

$$= S_n^2$$

(b) Assume WLOG that $\mathbb{E} X_1 = 0$. Set $\mathbb{E} X_1^j = \theta_j \quad j = 2, 3, 4.$

We have

$$\mathrm{Var}\left(S_n^2\right) = \mathbb{E}\left(S_n^4\right) - \left(\mathbb{E} S_n^2\right)^2,$$

where $\mathbb{E} S_n^2 = \sigma^2.$

We have

$$\mathbb{E}\, S_n^4 = \mathbb{E}\left(\frac{1}{2n(n-1)} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(X_j - X_k\right)^2\right)^2$$

$$= \frac{1}{4n^2(n-1)^2}\, \mathbb{E}\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \left(X_j - X_k\right)^2 \cdot \sum_{l=1}^{n} \sum_{m=1}^{n} \left(X_l - X_m\right)^2\right)$$

$$= \frac{1}{4n^2(n-1)^2}\Bigg[\underbrace{2n(n-1)\, \mathbb{E}\left(X_1 - X_2\right)^4}_{j\neq k,\ l\neq m,\ j=l,\ k=m \text{ or } j=m,\ k=l} + \underbrace{4\cdot n(n-1)(n-2)\, \mathbb{E}\left(X_1-X_2\right)^2\left(X_3-X_2\right)^2}_{\substack{j\neq k=l\neq m\neq j\\ j\neq k=m\neq l\neq j\\ k\neq j=l\neq m\neq k\\ k\neq j=m\neq l\neq k}}$$

$$\underbrace{n(n-1)(n-2)(n-3)\cdot \mathbb{E}\left(X_1-X_2\right)^2\left(X_3-X_4\right)^2}_{k,j,m,l \text{ all distinct}}\Bigg].$$

Now

$$\mathbb{E}\left(X_1 - X_2\right)^4 = \mathbb{E}\left[X_1^4 - 3X_1^3X_2 + 6X_1^2X_2^2 - 3X_1X_2^3 + X_2^4\right]$$

$$= \theta_4 + 6\,\theta_2^2 + \theta_4$$

$$= 2\left(\theta_4 + 3\,\theta_2^2\right),$$

$$\mathbb{E}\left(X_1-X_2\right)^2\left(X_3-X_2\right)^2 = \mathbb{E}\left[\left(X_1^2 - 2X_1X_2 + X_2^2\right)\left(X_3^2 - 2X_3X_2 + X_2^2\right)\right]$$

$$= \mathbb{E}\Big[X_1^2X_3^2 - 2X_1^2X_3X_2 + X_1^2X_2^2$$

$$-2X_1X_2X_3^2 + 4X_1X_3X_2^2 - 2X_1X_2^3$$

$$+ X_2^2X_3^2 - 2X_3X_2^3 + X_2^4\Big]$$

$$= \theta_2^2 + 0 + \theta_2^2 - 0 + 0 - + \theta_2^2 - 0 + \theta_4$$

$$= 3\theta_2^2 + \theta_4,$$

$$\mathbb{E}(X_1-X_2)^2(X_3-X_4)^2 = \mathbb{E}(X_1-X_2)^2\,\mathbb{E}(X_3-X_4)^2$$

$$= \mathbb{E}\left(X_1^2 - 2X_1X_2 + X_2^2\right)\mathbb{E}\left(X_3^2 - 2X_3X_4 + X_4^2\right)$$

$$= 2\theta_2 \cdot 2\theta_2$$

$$= 4\theta_2^2.$$

So we have

$$\mathbb{E}S_n^4 = \frac{1}{4n^2(n-1)^2}\Big[2n(n-1)\;2\cdot(\theta_4 + 3\theta_2^2) + 4\,n(n-1)(n-2)\left(3\theta_2^2 + \theta_4\right)$$
$$+ n(n-1)(n-2)(n-3)\;4\theta_2^2\Big]$$

$$= \frac{1}{n(n-1)}\Big[\theta_4 + 3\theta_2^2 + (n-2)\left(3\theta_2^2 + \theta_4\right) + (n-2)(n-3)\,\theta_2^2\Big]$$

$$= \frac{1}{n}\left[\theta_4\left(\frac{1}{n-1} + \frac{n-2}{n-1}\right) + \theta_2^2\left(\frac{3}{n-1} + \frac{3(n-2)}{n-1} + \frac{(n-2)(n-3)}{n-1}\right)\right]$$

$$= \frac{1}{n}\left[\theta_4 + \frac{\theta_2^2}{n-1}\left(3 + 3n - 6 + n^2 - 5n + 6\right)\right]$$

$$= \frac{1}{n}\left[\theta_4 + \frac{\theta_2^2}{n-1}\left(n^2 - 2n + 3\right)\right]$$

$$= \frac{1}{n}\left[\theta_4 - \frac{\theta_2^2}{n-1}\left(n - 3 - n^2 + n\right)\right]$$

$$= \frac{1}{n}\left[\theta_4 - \theta_2^2\left(\frac{n-3}{n-1}\right) + \frac{\theta_2^2}{n-1}\, n(n-1)\right]$$

$$= \frac{1}{n}\left[\theta_4 - \theta_2^2\left(\frac{n-3}{n-1}\right)\right] + \theta_2^2 .$$

Now, plugging this in, we get

$$\mathrm{Var}(S_n^2) = \mathbb{E}S_n^4 - (\mathbb{E}S_n^2)^2 = \frac{1}{n}\left[\theta_4 - \theta_2^2\left(\frac{n-3}{n-1}\right)\right].$$

(c) Find $\mathrm{Cov}(\bar{X}_n, S_n^2)$. Assume, WLOG, that $\mathbb{E}X_1 = 0$.

We have

$$\mathrm{Cov}(\bar{X}_n, S_n^2) = \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{2n(n-1)}\sum_{j=1}^n\sum_{k=1}^n (X_j - X_k)^2\right)$$

$$= \frac{1}{n}\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \mathrm{Cov}(X_i, (X_j - X_k)^2)$$

$$= \frac{1}{n}\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\left\{\mathrm{Cov}(X_i, (X_i - X_j)^2) + \mathrm{Cov}(X_i, (X_j - X_i)^2)\right\}$$

$$= \frac{1}{n}\frac{1}{2n(n-1)}\, n(n-1)\, 2\,\theta_3$$

$$= \frac{\theta_3}{n},$$ $\leftarrow$ So when $\theta_3 = 0$, $\mathrm{Cov}(\bar{X}_n, S_n^2) = 0$.

Since

$$\mathrm{Cov}(X_i, (X_j - X_k)^2) = \begin{cases} \mathrm{Cov}(X_1, (X_2 - X_1)^2) & \text{if } i = k \neq j \\ \mathrm{Cov}(X_1, (X_1 - X_2)^2) & \text{if } i = j \neq k \\ 0 & \text{o.w.,} \end{cases}$$

and

$$\mathrm{Cov}(X_1, (X_2 - X_1)^2) = \mathbb{E}X_1(X_2 - X_1)^2 = \mathbb{E}\,X_1(X_2^2 - 2X_1X_2 + X_1^2) = \mathbb{E}X_1^3 = \theta_3 .$$

$\boxed{5.10}$

(a) Let $X \sim \text{Normal}(\mu, \sigma^2)$. Then $\theta_1 = \mu$.

Now let $U = X - \mu$.

Then $U \sim \text{Normal}(0, \sigma^2)$ and $\theta_k = \mathbb{E}U^k$, $k = 2, 3, 4$.

The mgf of U is

$$M_U(t) = e^{\sigma^2 t^2/2}$$

$$\frac{d}{dt} M_U(t) = e^{\sigma^2 t^2/2} \sigma^2 t$$

$$\frac{d^2}{dt^2} M_U(t) = e^{\sigma^2 t^2/2} \sigma^2 t \cdot \sigma^2 t + \sigma^2 e^{\sigma^2 t^2/2}$$

$$= e^{\sigma^2 t^2/2} \sigma^2 (\sigma^2 t^2 + 1)$$

$$\frac{d^3}{dt^3} M_U(t) = e^{\sigma^2 t^2/2} \sigma^2 (\sigma^2 t)(\sigma^2 t^2 + 1) + e^{\sigma^2 t^2/2} \sigma^2 \sigma^2 2t$$

$$= e^{\sigma^2 t^2/2} \sigma^4 (\sigma^2 t^3 + 3t)$$

$$\frac{d^4}{dt^4} M_U(t) = e^{\sigma^2 t^2/2} \sigma^4 (\sigma^2 t)(\sigma^2 t^3 + 3t) + e^{\sigma^2 t^2/2} \sigma^4 (3\sigma^2 t^2 + 3)$$

$$= e^{\sigma^2 t^2/2} \sigma^4 \left(\sigma^4 t^4 + 3\sigma^2 t^2 + 3\sigma^2 t^2 + 3 \right)$$

From here we have

$$\theta_2 = M_U^{(2)}(0) = \sigma^2$$

$$\theta_3 = M_U^{(3)}(0) = 0$$

$$\theta_4 = M_U^{(4)}(0) = 3\sigma^4$$

(b) So we have

$$\operatorname{Var} S_n^2 = \frac{1}{n}\left[\theta_4 - \left(\frac{n-3}{n-1}\right)\theta_2^2\right]$$

$$= \frac{1}{n}\left[3\sigma^4 - \left(\frac{n-3}{n-1}\right)\sigma^4\right]$$

$$= \frac{1}{n}\left[\frac{(n-1)3\sigma^4 - (n-3)\sigma^4}{n-1}\right]$$

$$= \frac{2n\sigma^4}{n(n-1)}$$

$$= \frac{2\sigma^4}{n-1}$$

(c) Using the fact that

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1},$$

we have

$$\operatorname{Var}\left[\frac{(n-1)S_n^2}{\sigma^2}\right] = 2(n-1), \quad \left(\text{look up variance of } \chi^2_{n-1}\right)$$

where

$$\operatorname{Var}\left[\frac{(n-1)S_n^2}{\sigma^2}\right] = \frac{(n-1)^2}{\sigma^4}\operatorname{Var} S_n^2 .$$

Solving for $\operatorname{Var} S_n^2$ gives

$$\operatorname{Var} S_n^2 = \frac{2\sigma^4}{n-1} .$$

$\boxed{5.12}$ Let $X_1, \ldots, X_n \overset{ind}{\sim} \text{Normal}(0,1)$. Let

$$Y_1 = \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|, \qquad Y_2 = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

We have $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \sim \text{Normal}(0, \frac{1}{n})$, so $\sqrt{n}\, \bar{X}_n \sim \text{Normal}(0,1)$

For $Z \sim \text{Normal}(0,1)$,

$$\mathbb{E}|Z| = \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= 2 \int_0^{\infty} \frac{z}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$u = \frac{z^2}{2} \iff \sqrt{2u} = z$

$$= 2 \int_0^{\infty} \frac{\sqrt{2u}}{\sqrt{2\pi}} e^{-u} \frac{1}{\sqrt{2u}} \, du$$

$dz = \frac{1}{2\sqrt{2u}} 2 \, du = \frac{1}{\sqrt{2u}}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-u} \, du$$

$$= \sqrt{\frac{2}{\pi}} .$$

So

$$\mathbb{E}\, Y_1 = \mathbb{E}\, |\bar{X}_n| = \frac{1}{\sqrt{n}} \mathbb{E}\, |\sqrt{n}\, \bar{X}_n| = \frac{1}{\sqrt{n}} \mathbb{E}|Z| = \frac{1}{\sqrt{n}} \sqrt{\frac{2}{\pi}} .$$

Also

$$\mathbb{E}\, Y_2 = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} |X_i| \right) = \mathbb{E}|X_1| = \mathbb{E}|Z| = \sqrt{\frac{2}{\pi}} .$$

So we have $\mathbb{E}\, Y_1 \leq \mathbb{E}\, Y_2 \quad \forall \; n \geq 1$.

$\boxed{5.13}$ Let $X_1, \dots, X_n \overset{ind}{\sim} \text{Normal}(\mu, \sigma^2)$. Let $W = \frac{(n-1)S_n^2}{\sigma^2}$.

We know $W \sim \chi^2_{n-1}$. Let's find $\mathbb{E}\sqrt{\frac{\sigma^2}{(n-1)} W} = \mathbb{E} S_n$.

We have

$$\mathbb{E} S_n = \int_0^\infty \sqrt{\frac{\sigma^2}{n-1} w} \, \frac{1}{\Gamma(\frac{n-1}{2}) 2^{\frac{n-1}{2}}} w^{\frac{n-1}{2}-1} e^{-w/2} \, dw$$

$$= \frac{\sigma}{\sqrt{n-1}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \frac{2^{n/2}}{2^{\frac{n-1}{2}}} \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{n}{2}) 2^{n/2}} w^{\frac{n}{2}-1} e^{-w/2} \, dw}_{=1}$$

$$= \frac{\sigma}{\sqrt{n-1}} \frac{\sqrt{2}\,\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}.$$

So set $$g(S_n^2) = \sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{S_n^2}.$$

Then $$\mathbb{E}\, g(S_n^2) = \sigma.$$

$\boxed{5.16}$ Let $X_1 \sim N(1,1)$, $X_2 \sim N(2,4)$, $X_3 \sim N(3,9)$ be indep.

(a) $$(X_1 - 1)^2 + \left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2 \sim \chi^2_3$$

(b) $$\frac{(X_1 - 1)}{\sqrt{\left\{\left(\frac{X_2-2}{2}\right)^2 + \left(\frac{X_3-3}{3}\right)^2\right\}/2}} \sim t_2$$

(c) $$\frac{(X_1 - 1)^2}{\left\{\left(\frac{X_2-2}{2}\right)^2 + \left(\frac{X_3-3}{3}\right)^2\right\}/2} \sim F_{1,2}$$

$\boxed{5.17}$ Let $W_1 \sim \chi^2_{\nu_1}$ and $W_2 \sim \chi^2_{\nu_2}$ be indep rvs.

It can be shown (previous hw) that

$$X = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1,\nu_2}$$

(b) Using independence of W_1 and W_2 and the fact $\mathbb{E}W_1 = \nu_1$, we have

$$\mathbb{E}X = \mathbb{E}\left[\frac{W_1/\nu_1}{W_2/\nu_2}\right] = \frac{\nu_2}{\nu_1}\mathbb{E}W_1 \cdot \mathbb{E}\frac{1}{W_2} = \frac{\nu_2}{\nu_1}\nu_1 \cdot \mathbb{E}\frac{1}{W_2} = \nu_2\,\mathbb{E}\frac{1}{W_2}.$$

Now

$$\mathbb{E}\frac{1}{W_2} = \int_0^\infty \frac{1}{w}\frac{1}{\Gamma(\frac{\nu_2}{2})2^{\nu_2/2}} w^{\frac{\nu_2}{2}-1} e^{-w/2}\,dw$$

$$= \frac{\Gamma(\frac{\nu_2-2}{2})2^{\frac{\nu_2-2}{2}}}{\Gamma(\frac{\nu_2}{2})2^{\nu_2/2}} \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{\nu_2-2}{2})2^{\frac{\nu_2-2}{2}}} w^{\frac{\nu_2-2}{2}-1} e^{-w/2}\,dw}_{=1,\ \text{provided } \nu_2>2}$$

$$= \frac{\Gamma(\frac{\nu_2-2}{2})}{\frac{\nu_2-2}{2}\Gamma(\frac{\nu_2-2}{2})\,2}$$

$$= \frac{1}{\nu_2-2}.$$

So we have

$$\mathbb{E}X = \frac{\nu_2}{\nu_2-2}, \quad \text{provided } \nu_2 > 2.$$

Now we have $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}\left[\frac{W_1^2/\nu_1^2}{W_2^2/\nu_2^2}\right] - \left(\frac{\nu_2}{\nu_2-2}\right)^2.$

For the first term, using independence of W_1 and W_2 and

$$\mathbb{E} W_1^2 = \operatorname{Var} W_1 + (\mathbb{E} W_1)^2 = 2\nu_1 + \nu_1^2 = \nu_1(2+\nu_1),$$

we obtain

$$\mathbb{E}\left[\frac{W_1^2/\nu_1^2}{W_2^2/\nu_2^2}\right] = \frac{\nu_2^2}{\nu_1^2}\,\mathbb{E} W_1^2 \cdot \mathbb{E}\frac{1}{W_2^2} = \frac{\nu_2^2}{\nu_1}(2+\nu_1)\,\mathbb{E}\frac{1}{W_2^2}.$$

Now we have

$$\mathbb{E}\frac{1}{W_2^2} = \int_0^\infty \frac{1}{w^2}\frac{1}{\Gamma\left(\frac{\nu_2}{2}\right)2^{\nu_2/2}} w^{\nu_2/2-1} e^{-w/2}\,dw$$

$$= \frac{\Gamma\left(\frac{\nu_2-4}{2}\right)2^{\frac{\nu_2-4}{2}}}{\Gamma\left(\frac{\nu_2}{2}\right)2^{\nu_2/2}} \underbrace{\int_0^\infty \frac{1}{\Gamma\left(\frac{\nu_2-4}{2}\right)2^{\frac{\nu_2-4}{2}}} w^{\frac{\nu_2-4}{2}-1} e^{-w/2}\,dw}_{=1 \text{ provided } \nu_2>4.}$$

$$= \frac{1}{\left(\frac{\nu_2-2}{2}\right)\left(\frac{\nu_2-4}{2}\right)2^2}$$

$$= \frac{1}{(\nu_2-2)(\nu_2-4)}$$

So

$$\mathbb{E}X^2 = \frac{\nu_2^2}{\nu_1}(2+\nu_1)\frac{1}{(\nu_2-2)(\nu_2-4)}$$

and

$$\operatorname{Var} X = \frac{\nu_2^2(2+\nu_1)}{\nu_1(\nu_2-2)(\nu_2-4)} - \left(\frac{\nu_2}{\nu_2-2}\right)^2$$

$\swarrow$ $2\nu_2 - 4 + \nu_1\nu_2 - 2\nu_1 - \nu_1\nu_2 + 4\nu_1 = 2(\nu_1 + \nu_2 - 2)$

$$= \frac{\nu_2^2 (2+\nu_1)(\nu_2-2) - \nu_2^2 \nu_1 (\nu_2 - 4)}{\nu_1 (\nu_2-2)^2 (\nu_2-4)}$$

$$= 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$$

(c) We have

$$\frac{1}{X} = \frac{W_2/\nu_2}{W_1/\nu_1} \sim F_{\nu_2,\nu_1}.$$

(d) Let $Y = \dfrac{\left(\frac{\nu_1}{\nu_2}\right)X}{1+\left(\frac{\nu_1}{\nu_2}\right)X}$. Then Y has support on $(0,1)$.

Also,
$$y = \frac{(\nu_1/\nu_2)\,x}{1+\left(\frac{\nu_1}{\nu_2}\right)x} = g(x) \iff y = \frac{(\nu_1/\nu_2)}{\frac{1}{x} + (\nu_1/\nu_2)}$$

$$\iff \frac{1}{x} + (\nu_1/\nu_2) = \frac{(\nu_1/\nu_2)}{y}$$

$$\iff \frac{1}{x} = \frac{(\nu_1/\nu_2)}{y} - (\nu_1/\nu_2) = \left(\frac{\nu_1}{\nu_2}\right)\left(\frac{1}{y} - 1\right)$$

$$\iff x = \left(\frac{\nu_2}{\nu_1}\right)\left(\frac{y}{1-y}\right) = g^{-1}(y).$$

So
$$\frac{d}{dy} g^{-1}(y) = \frac{\nu_2}{\nu_1}\left[\frac{y}{(1-y)^2} + \frac{1}{1-y}\right] = \frac{\nu_2}{\nu_1}\left(\frac{1}{1-y}\right)^2$$

The pdf of X is given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1-2}{2}} \left(1 + \frac{\nu_1}{\nu_2}x\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} \mathbb{1}(x>0),$$

and the transformation method gives, for $y \in (0,1)$,

$$f_Y(y) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \left[\frac{\nu_2}{\nu_1}\left(\frac{y}{1-y}\right)\right]^{\frac{\nu_1-2}{2}} \left(1 + \frac{\nu_1}{\nu_2}\frac{\nu_2}{\nu_1}\left(\frac{y}{1-y}\right)\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} \left| -\frac{\nu_2}{\nu_1}\left(\frac{1}{1-y}\right)^2 \right|$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_1}{2}} y^{\frac{\nu_1}{2}-1} (1-y)^{-\frac{\nu_1}{2}+1-2} (1-y)^{\frac{\nu_1+\nu_2}{2}}$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} y^{\frac{\nu_1}{2}-1} (1-y)^{\frac{\nu_2}{2}-1},$$

which is the pdf of the Beta$\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$ distribution.