STAT 712 hw 9

Order statistics, convergence in probability

Do problems 5.23, 5.24, 5.27(a) from CB. In addition:

1. For a random sample $X_1, \ldots, X_n \overset{\text{ind}}{\sim} f_X$, the joint density of all the order statistics $X_{(1)} < \cdots < X_{(n)}$ is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \cdot \prod_{i=1}^{n} f_X(x_i) \cdot \mathbf{1}(-\infty < x_1 < \cdots < x_n < \infty).$$

Suppose $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(1)$, with order statistics $X_{(1)} < \cdots < X_{(n)}$, and let

$$\begin{aligned} Z_1 &= nX_{(1)} \\ Z_2 &= (n-1)(X_{(2)} - X_{(1)}) \\ Z_3 &= (n-2)(X_{(3)} - X_{(2)}) \\ &\vdots \\ Z_n &= X_{(n)} - X_{(n-1)}. \end{aligned}$$

   (a) Find the joint pdf of $Z_1, \ldots, Z_n$.

   (b) State whether $Z_1, \ldots, Z_n$ are mutually independent.

   (c) Give the marginal distribution of each of $Z_1, \ldots, Z_n$.

2. For a random sample $X_1, \ldots, X_n$ from a continuous distribution with median M, give

$$P(X_{(1+l)} \leq M \leq X_{(n-l)})$$

for each integer $0 \leq l < \frac{n-1}{2}$. Evaluate the probability for $n = 15$ and $l = 3$. The idea is that we can choose l so the interval $(X_{(1+l)}, X_{(n-l)})$ will contain M with some desired probability (this is a confidence interval for the median). *Hint: When $n = 3$ and $l = 0$ the probability is 0.75.*

3. (Optional) Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\mu - \theta, \mu + \theta)$, $n \geq 2$, and consider the sequences of random variables $\{R_n\}_{n \geq 2}$ and $\{M_n\}_{n \geq 2}$ given by

$$R_n = \frac{X_{(n)} - X_{(1)}}{2} \quad \text{and} \quad M_n = \frac{X_{(1)} + X_{(n)}}{2}$$

for $n \geq 2$.

   (a) Find the joint pdf of (R_n, M_n).

   (b) Find the marginal pdf of R_n.

   (c) Find the marginal pdf of M_n.

   (d) Show that R_n converges in probability to θ as $n \to \infty$.

   (e) Show that M_n converges in probability to μ as $n \to \infty$.