STAT 712 hw 10

Convergence in distribution, central limit theorem, Slutzky's, delta method

Do problems 5.18 ~~(a)~~, 5.30, 5.36, 5.44, 5.51 (a),(b) from CB. In addition:

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$. Show that the interval $\sqrt{\bar{X}_n} \pm z_{\alpha/2}/(2\sqrt{n})$ contains $\sqrt{\lambda}$ with probability tending to $1 - \alpha$ as $n \to \infty$.

2. A real-valued function g is *uniformly continuous on* $\mathcal{A}$ if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $x, x' \in \mathcal{A}$ and $|x' - x| < \delta_\varepsilon$ then $|g(x') - g(x)| < \varepsilon$. Let the random variables $\{X_n\}_{n \geq 1}$ and X have support on $\mathcal{X}$ and suppose g is uniformly continuous on $\mathcal{X}$. Show that $g(X_n) \xrightarrow{\text{p}} g(X)$ if $X_n \xrightarrow{\text{p}} X$.

3. (Optional) Let $\theta_1, \dots, \theta_n$ be independent realizations of the random variable θ, for which we have

$$\begin{aligned} \mathbb{E}\cos\theta &= \rho\cos\mu \\ \mathbb{E}\sin\theta &= \rho\sin\mu \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{E}\cos(2(\theta-\mu)) &= \alpha_2 \\ \mathbb{E}\sin(2(\theta-\mu)) &= \beta_2 \end{aligned}$$

   Define estimators $\hat{\rho}$ and $\hat{\mu}$ by the equations $\hat{\rho}\cos\hat{\mu} = n^{-1}\sum_{i=1}^n \cos\theta_i$ and $\hat{\rho}\sin\hat{\mu} = n^{-1}\sum_{i=1}^n \sin\theta_i$. This question is inspired by the paper [1]. The setting is circular data, in which angles or directions $\theta_1, \dots, \theta_n$ are observed and one wishes to estimate the mean angle μ.

   (a) Show that $\sqrt{n}(\hat{\rho}\cos(\hat{\mu} - \mu) - \rho) \xrightarrow{\text{D}} \text{Normal}(0, (1 + \alpha_2 - 2\rho^2)/2)$ as $n \to \infty$.
   Hint: Show $\hat{\rho}\cos(\hat{\mu} - \mu) = n^{-1}\sum_{i=1}^n \cos(\theta_i - \mu)$ *and find* $\text{Var}\cos(\theta - \mu)$.

   (b) Show that $n(1 - \cos(\hat{\mu} - \mu))/\sigma^2 \xrightarrow{\text{D}} \chi_1^2$ as $n \to \infty$, where $\sigma^2 = (1 - \alpha_2)/(4\rho^2)$.
   This takes some time. Focus on the rest of the hw first.

   (c) It can be shown that $\hat{\sigma}^2 = [1 - n^{-1}\sum_{i=1}^n \cos(2(\theta_i - \hat{\mu}))]/(4\hat{\rho}^2)$ is a consistent estimator of σ^2. Use this fact to argue that the interval $\hat{\mu} \pm \cos^{-1}(1 - \hat{\sigma}^2\chi_{1,\alpha}^2/n)$ will contain μ with probability approaching $1 - \alpha$ as $n \to \infty$.

4. (Optional) Additional problems from CB: 5.35, 5.42

References

[1] Nicholas I Fisher and Peter Hall. Bootstrap confidence regions for directional data. Journal of the American Statistical Association, 84(408):996–1002, 1989.