STAT 712 hw 10

Convergence in distribution, central limit theorem, Slutzky's, delta method

Do problems 5.18 ~~(a)~~, 5.30, 5.44, 5.51 (a),(b) from CB. In addition:

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$. Show that the interval $\sqrt{\bar{X}_n} \pm z_{\alpha/2}/(2\sqrt{n})$ contains $\sqrt{\lambda}$ with probability tending to $1 - \alpha$ as $n \to \infty$.

By the central limit theorem we have

$$\sqrt{n}(\bar{X}_n - \lambda) \xrightarrow{\text{D}} \text{Normal}(0, \lambda),$$

since the mean and variance of the $\text{Poisson}(\lambda)$ distribution are both equal to λ. Now, using the delta method with the function $g(z) = \sqrt{z}$, of which the derivative is $g'(z) = 1/(2\sqrt{z})$, we have

$$\sqrt{n}(\sqrt{\bar{X}_n} - \sqrt{\lambda}) \xrightarrow{\text{D}} \text{Normal}\left(0, \frac{1}{4}\right),$$

where the asymptotic variance comes from

$$[g'(\lambda)]^2 \cdot \lambda = \left(\frac{1}{2\sqrt{\lambda}}\right)^2 \cdot \lambda = \frac{1}{4}.$$

We see that

$$2\sqrt{n}(\sqrt{\bar{X}_n} - \sqrt{\lambda}) \xrightarrow{\text{D}} \text{Normal}(0, 1),$$

so that

$$\lim_{n\to\infty} P\left(-z_{\alpha/2} < 2\sqrt{n}(\sqrt{\bar{X}_n} - \sqrt{\lambda}) < z_{\alpha/2}\right) = 1 - \alpha.$$

Rearranging the above gives the upper and lower bounds of the confidence interval.

2. A real-valued function g is *uniformly continuous on* $\mathcal{A}$ if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $x, x' \in \mathcal{A}$ and $|x' - x| < \delta_\varepsilon$ then $|g(x') - g(x)| < \varepsilon$. Let the random variables $\{X_n\}_{n \geq 1}$ and X have support on $\mathcal{X}$ and suppose g is uniformly continuous on $\mathcal{X}$. Show that $g(X_n) \xrightarrow{\text{p}} g(X)$ if $X_n \xrightarrow{\text{p}} X$.

Choose $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that $|X_n - X| < \delta_\varepsilon \implies |g(X_n) - g(X)| < \varepsilon$; that is $\{|X_n - X| < \delta_\varepsilon\} \subset \{|g(X_n) - g(X)| < \varepsilon\}$. Therefore

$$P(|g(X_n) - g(X)| < \varepsilon) \geq P(|X_n - X| < \delta_\varepsilon).$$

The right side goes to 1 as $n \to \infty$, giving $\lim_{n\to\infty} P(|g(X_n) - g(X)| < \varepsilon) = 1$. Since $\varepsilon > 0$ was chosen arbitrarily, we have shown $g(X_n) \xrightarrow{\text{p}} g(X)$.

3. (Optional) Let $\theta_1, \ldots, \theta_n$ be independent realizations of the random variable θ, for which we have

$$\begin{aligned} \mathbb{E}\cos\theta &= \rho\cos\mu \\ \mathbb{E}\sin\theta &= \rho\sin\mu \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbb{E}\cos(2(\theta-\mu)) &= \alpha_2 \\ \mathbb{E}\sin(2(\theta-\mu)) &= \beta_2 \end{aligned}$$

Define estimators $\hat{\rho}$ and $\hat{\mu}$ by the equations $\hat{\rho}\cos\hat{\mu} = n^{-1}\sum_{i=1}^n \cos\theta_i$ and $\hat{\rho}\sin\hat{\mu} = n^{-1}\sum_{i=1}^n \sin\theta_i$. This question is inspired by the paper [1]. The setting is circular data, in which angles or directions $\theta_1, \ldots, \theta_n$ are observed and one wishes to estimate the mean angle μ.

(a) Show that $\sqrt{n}(\hat{\rho}\cos(\hat{\mu}-\mu) - \rho) \xrightarrow{\text{D}} \text{Normal}(0, (1+\alpha_2-2\rho^2)/2)$ as $n \to \infty$.
Hint: Show $\hat{\rho}\cos(\hat{\mu}-\mu) = n^{-1}\sum_{i=1}^n \cos(\theta_i - \mu)$ *and find* $\text{Var}\cos(\theta-\mu)$.

Using the trigonometric identity $\cos(x-y) = \cos x\cos y - \sin x \sin y$, we may write

$$\begin{aligned} \hat{\rho}\cos(\hat{\mu}-\mu) &= \hat{\rho}\cos\hat{\mu}\cos\mu + \hat{\rho}\sin\hat{\mu}\sin\mu \\ &= (n^{-1}\textstyle\sum_{i=1}^n \cos\theta_i)\cos\mu + (n^{-1}\sum_{i=1}^n \sin\theta_i)\sin\mu \\ &= n^{-1}\textstyle\sum_{i=1}^n (\cos\theta_i\cos\mu + \sin\theta_i\sin\mu) \\ &= n^{-1}\textstyle\sum_{i=1}^n \cos(\theta_i - \mu). \end{aligned}$$

Moreover, making use of the trigonometric identities

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \cos(x-y) = \cos x\cos y - \sin x\sin y,$$

we have

$$\begin{aligned} \text{Var}\cos(\theta-\mu) &= \mathbb{E}\cos^2(\theta-\mu) - (\mathbb{E}\cos(\theta-\mu))^2 \\ &= \frac{1}{2}\mathbb{E}[1+\cos(2(\theta-\mu))] - [\mathbb{E}(\cos\theta\cos\mu + \sin\theta\sin\mu)]^2 \\ &= \frac{1}{2}(1+\alpha_2) - [\rho\cos^2\mu + \rho\sin^2\mu]^2 \\ &= \frac{1}{2}(1+\alpha_2 - 2\rho^2). \end{aligned}$$

From here the central limit theorem gives the result.

(b) Show that $n(1-\cos(\hat{\mu}-\mu))/\sigma^2 \xrightarrow{\text{D}} \chi_1^2$ as $n \to \infty$, where $\sigma^2 = (1-\alpha_2)/(4\rho^2)$.
This takes some time. Focus on the rest of the hw first.

Letting $X_i = \cos(\theta_i - \mu)$ and $Y_i = \sin(\theta_i - \mu)$ for $i = 1, \ldots, n$ and setting $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\bar{Y}_n = \sum_{i=1}^n Y_i$, we can show that $\cos(\hat{\mu}-\mu) = g(\bar{X}_n, \bar{Y}_n)$ for $g(x,y) = x/\sqrt{x^2+y^2}$. First write

$$\begin{aligned}\hat\rho\sin(\hat\mu-\mu) &= \hat\rho\sin\hat\mu\cos\mu - \hat\rho\sin\hat\mu\cos\mu\\ &= (n^{-1}\textstyle\sum_{i=1}^n \sin\theta_i)\cos\mu - (n^{-1}\sum_{i=1}^n \sin\theta_i)\cos\mu\\ &= n^{-1}\textstyle\sum_{i=1}^n(\sin\theta_i\cos\mu + \sin\theta_i\cos\mu)\\ &= n^{-1}\textstyle\sum_{i=1}^n \sin(\theta_i-\mu).\end{aligned}$$

From this and from our earlier work we may write

$$\begin{aligned}\hat\rho^2 &= \hat\rho^2(\sin^2(\hat\mu-\mu)+\cos^2(\hat\mu-\mu))\\ &= (\hat\rho\cos(\hat\mu-\mu))^2 + (\hat\rho\sin(\hat\mu-\mu))^2\\ &= (n^{-1}\textstyle\sum_{i=1}^n\cos(\theta_i-\mu))^2 + (n^{-1}\sum_{i=1}^n\sin(\theta_i-\mu))^2\\ &= \bar X_n^2 + \bar Y_n^2.\end{aligned}$$

This gives

$$\cos(\hat\mu-\mu) = \frac{\hat\rho\cos(\hat\mu-\mu)}{\hat\rho} = \frac{\bar X_n}{\sqrt{\bar X_n^2+\bar Y_n^2}}.$$

Now

$$\mathbb{E}X_1 = \mathbb{E}\cos(\theta_1-\mu) = \mathbb{E}(\cos\theta_1\cos\mu + \sin\theta_1\sin\mu) = \rho\cos^2\mu + \rho\sin^2\mu = \rho,$$

and

$$\begin{aligned}\mathbb{E}Y_1 &= \mathbb{E}\sin(\theta_1-\mu)\\ &= \mathbb{E}(\sin\theta_1\cos\mu + \sin\theta_1\cos\mu)\\ &= \rho\sin\mu\cos\mu + \rho\sin\mu\cos\mu\\ &= \rho\sin(\mu-\mu)\\ &= 0,\end{aligned}$$

where we have used the trigonometric identity $\sin(x-y) = \sin x\cos x - \sin y\cos y$. So

$$g(\mu_X,\mu_Y) = g(\rho,0) = \rho/(\rho^2+0)^{1/2} = 1.$$

If we try the first-order delta method on $\sqrt{n}(\cos(\hat\mu-\mu)-1) = \sqrt{n}(g(\bar X_n,\bar Y_n) - g(\mu_X,\mu_Y))$, we run into a problem: For the function $g(x,y) = x/\sqrt{x^2+y^2}$ we have

$$\begin{aligned}\frac{\partial}{\partial x}g(x,y) &= \frac{y^2}{(x^2+y^2)^{3/2}}\\ \frac{\partial}{\partial y}g(x,y) &= -\frac{xy}{(x^2+y^2)^{3/2}},\end{aligned}$$

each of which is equal to zero when we plug in $(x, y) = (\rho, 0)$. Therefore we cannot use the first-order delta method.

Since we cannot use the first-order delta method, we establish the result with the second-order expansion

$$g(\bar{X}_n, \bar{Y}_n) \approx g(\mu_X, \mu_Y) + \dot{\mathbf{g}}(\mu_X, \mu_Y)^T \begin{pmatrix} \bar{X}_n - \mu_X \\ \bar{Y}_n - \mu_Y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \bar{X}_n - \mu_X \\ \bar{Y}_n - \mu_Y \end{pmatrix}^T \ddot{\mathbf{g}}(\mu_X, \mu_Y) \begin{pmatrix} \bar{X}_n - \mu_X \\ \bar{Y}_n - \mu_Y \end{pmatrix},$$

where

$$\dot{\mathbf{g}}(\mu_X, \mu_Y) = \begin{pmatrix} \frac{\partial}{\partial x} g(x,y) \\ \frac{\partial}{\partial y} g(x,y) \end{pmatrix} \Bigg|_{(x,y)=(\mu_X,\mu_Y)}, \quad \ddot{\mathbf{g}}(\mu_X, \mu_Y) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} g(x,y) & \frac{\partial^2}{\partial x \partial y} g(x,y) \\ \frac{\partial^2}{\partial x \partial y} g(x,y) & \frac{\partial^2}{\partial y^2} g(x,y) \end{pmatrix} \Bigg|_{(x,y)=(\mu_X,\mu_Y)}.$$

After some more differentiation of the function $g(x, y)$, we obtain

$$\ddot{\mathbf{g}}(\rho, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -\rho^{-2} \end{bmatrix},$$

which, with $g(\rho, 0) = 1$ and $\dot{\mathbf{g}}(\rho, 0) = 0$, gives the approximation

$$g(\bar{X}_n, \bar{Y}_n) \approx 1 - \frac{1}{2} \cdot \frac{(\bar{Y}_n - \mu_Y)^2}{2\rho^2} = 1 - \frac{\sigma_Y^2}{2n\rho^2} \left(\frac{\sqrt{n}(\bar{Y}_n - \mu_Y)}{\sigma_Y} \right)^2,$$

where $\sigma_Y^2 = \operatorname{Var} Y_1$. We now use the fact that $\sqrt{n}(\bar{Y}_n - \mu_Y)/\sigma_Y$ converges in distribution to a $\text{Normal}(0, 1)$ random variable, which when squared has the χ_1^2 distribution we can represent this random variable with W. Lastly, we plug in

$$\begin{aligned} \sigma_Y^2 &= \operatorname{Var} \sin(\theta - \mu) \\ &= \mathbb{E} \sin^2(\theta - \mu) - [\mathbb{E} \sin(\theta - \mu)]^2 \\ &= \frac{1}{2} \mathbb{E}(1 - \cos(2(\theta - \mu)) - (0)^2 \\ &= \frac{1 - \alpha_2}{2}, \end{aligned}$$

and rearrange to obtain

$$n[1 - \cos(\hat{\mu} - \mu)] \approx \frac{1 - \alpha_2}{4\rho^2} \cdot W.$$

(c) It can be shown that $\hat{\sigma}^2 = [1 - n^{-1} \sum_{i=1}^n \cos(2(\theta_i - \hat{\mu}))]/(4\hat{\rho}^2)$ is a consistent estimator of σ^2. Use this fact to argue that the interval $\hat{\mu} \pm \cos^{-1}(1 - \hat{\sigma}^2 \chi^2_{1,\alpha}/n)$ will contain μ with probability approaching $1 - \alpha$ as $n \to \infty$.

We begin by writing $P(n[1-\cos(\hat{\mu}-\mu)]/\hat{\sigma}^2 < \chi^2_{1,\alpha}) \to 1-\alpha$ as $n \to \infty$. Then we have

$$\begin{aligned} \{n[1-\cos(\hat{\mu}-\mu)]/\hat{\sigma}^2 < \chi^2_{1,\alpha}\} &= \{\cos(\hat{\mu}-\mu) > 1 - \frac{\hat{\sigma}^2}{n}\chi^2_{1,\alpha}\} \\ &= \{|\hat{\mu}-\mu| < \cos^{-1}(1-\frac{\hat{\sigma}^2}{n}\chi^2_{1,\alpha})\}, \end{aligned}$$

so that $\hat{\mu}$ will lie within $\cos^{-1}(1-\frac{\hat{\sigma}^2}{n}\chi^2_{1,\alpha})$ of μ with probability approaching $1-\alpha$ as $n \to \infty$.

4. (Optional) Additional problems from CB: 5.35, 5.42

References

[1] Nicholas I Fisher and Peter Hall. Bootstrap confidence regions for directional data. Journal of the American Statistical Association, 84(408):996–1002, 1989.

Problems 5.18, 5.30, 5.36, 5.44, 5.51 (a), (b) from CB.

$\boxed{5.18}$ Let $X \sim t_\nu$, so

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1 + \frac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$

(a) Already done in previous hw.

(b) Let $R = X^2$. We can show $R \sim F_{1,\nu}$ in two ways:

(i) Since $X \sim t_\nu$, we can write

$$X = \frac{Z}{\sqrt{W/\nu}},$$

where $Z \sim N(0,1)$, $W \sim \chi^2_\nu$, and $Z \perp\!\!\!\perp W$.

Then

$$R = X^2 = \frac{Z^2/1}{W/\nu} \sim F_{1,\nu},$$

from "the anatomy" of an F distributed random variable.

(ii) We can use the cdf method: For $r > 0$ we have

$$\begin{aligned} P(R \le r) &= P(X^2 \le r) \\ &= P(-\sqrt{r} \le X \le \sqrt{r}) \\ &= F_X(\sqrt{r}) - F_X(-\sqrt{r}), \end{aligned}$$

so

$$f_R(r) = f_X(\sqrt{r})\left(\frac{1}{2\sqrt{r}}\right) - f_X(-\sqrt{r})\left(-\frac{1}{2\sqrt{r}}\right)$$

$$= \frac{1}{\sqrt{r}} f_X(\sqrt{r})$$

$$= \frac{1}{\sqrt{r}} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \frac{1}{\sqrt{v\pi}} \frac{1}{\left(1 + \frac{r}{v}\right)^{\frac{v+1}{2}}}$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{v}\right)^{\frac{1}{2}} \frac{r^{\frac{1-2}{2}}}{\left(1 + \frac{r}{v}\right)^{\frac{v+1}{2}}} .$$

We recognize this as the pdf of the $F_{1,v}$ distribution.

(c) We have

$$\lim_{v\to\infty} f(x;v) = \lim_{v\to\infty} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \frac{1}{\sqrt{v\pi}} \frac{1}{\left(1 + \frac{x^2}{v}\right)^{\frac{v+1}{2}}}$$

$$= \underbrace{\left[\lim_{v\to\infty} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\frac{v}{2}}}\right]}_{=1} \underbrace{\frac{1}{\sqrt{2\pi}} \left[\lim_{v\to\infty} \frac{1}{\left(1 + \frac{x^2/2}{v/2}\right)^{v/2}}\right]}_{\frac{1}{\sqrt{2\pi}} \frac{1}{e^{x^2/2}}} \underbrace{\left[\lim_{v\to\infty} \frac{1}{\left(1+\frac{x^2}{v}\right)^{1/2}}\right]}_{=1}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

To see the above: Stirling's formula gives

$$\Gamma(n+1) \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

for large n, so we write

$$\lim_{\nu \to \infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\frac{\nu}{2}}} = \lim_{\nu \to \infty} \frac{\sqrt{2\pi}\left(\frac{\nu-1}{2}\right)^{\left(\frac{\nu-1}{2}\right)+\frac{1}{2}} e^{-\left(\frac{\nu-1}{2}\right)}}{\sqrt{2\pi}\left(\frac{\nu-2}{2}\right)^{\left(\frac{\nu-2}{2}\right)+\frac{1}{2}} e^{-\left(\frac{\nu-2}{2}\right)}} \frac{1}{\sqrt{\frac{\nu}{2}}}$$

$$= e^{-1/2} \lim_{\nu \to \infty} \frac{\left(\frac{\nu-1}{2}\right)^{\nu/2}}{\left(\frac{\nu-2}{2}\right)^{\nu/2}} \frac{1}{\left(\frac{\nu-2}{2}\right)^{-1/2}\sqrt{\frac{\nu}{2}}}$$

$$= e^{-1/2} \underbrace{\lim_{\nu \to \infty} \left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}}}_{=1} \underbrace{\lim_{\nu \to \infty} \left(\frac{\nu-1}{\nu-2}\right)^{\nu/2}}_{e^{1/2}} .$$

$$= 1,$$

where

$$\left(\frac{\nu-1}{\nu-2}\right)^{\nu/2} = \left(\frac{(\nu-1)-1}{\nu-1}\right)^{-\nu/2}$$

$$= \left(1 - \frac{1}{\nu-1}\right)^{-\frac{\nu}{2}}$$

$$= \left[\left(1 - \frac{1}{\nu-1}\right)^{(\nu-1)}\right]^{-\frac{1}{2}} \left(1 - \frac{1}{\nu-1}\right)^{-1/2}$$

$$\to \left(e^{-1}\right)^{-1/2} \cdot 1 = e^{1/2}$$

(d) Since $X \xrightarrow{d} Z \sim \text{Normal}(0,1)$, we have

$$X^2 \xrightarrow{d} Z^2 \sim \chi^2_1$$

by a result called the continuous mapping theorem.

(e) Let $W_1 \sim \chi^2_q$ and $W_2 \sim \chi^2_p$ be independent rvs. Then

$$R = q\,\frac{W_1/q}{W_2/p} = \frac{W_1}{W_2/p} \xrightarrow{D} W_1 \sim \chi^2_1,$$

by Slutzky's theorem, since

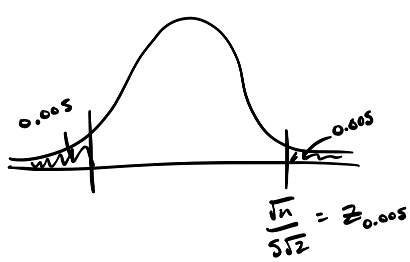
$$\frac{W_2}{p} = \frac{Z_1^2 + \cdots + Z_p^2}{p} \xrightarrow{p} 1, \quad \text{as } p \to \infty.$$

$\boxed{5.30}$ For large n we have

$$\bar{X}_1 - \bar{X}_2 \overset{\text{approx}}{\sim} \text{Normal}\left(0, \frac{2\sigma^2}{n}\right)$$

So

$$P\left(|\bar{X}_1 - \bar{X}_2| < \frac{\sigma}{5}\right) = P\left(\left|\frac{\bar{X}_1 - \bar{X}_2}{\sigma\sqrt{2/n}}\right| < \frac{1}{5\sqrt{2/n}}\right)$$



$$= P\left(|Z| < \frac{\sqrt{n}}{5\sqrt{2}}\right), \quad Z \sim N(0,1).$$

$$= 0.99$$

$$\iff \quad \frac{\sqrt{n}}{5\sqrt{2}} = \underbrace{z_{0.005}}_{=\frac{0.01}{2}} = \text{qnorm}(.995) = 2.576$$

$$\iff \quad n = (5 \cdot \sqrt{2} \cdot z_{0.005})^2 = (5 \cdot \sqrt{2} \cdot 2.576)^2 = 331.7$$

So take $n = 332$

$\boxed{5.44}$ Let $X_1, \ldots, X_n \overset{ind}{\sim}$ Bernoulli(p), $Y_n = n^{-1} \sum_{i=1}^{n} X_i$.

(a) The central limit theorem gives

$$\sqrt{n}\,(Y_n - p) \xrightarrow{D} N(0, p(1-p)) \quad \text{as} \quad n \to \infty,$$

since $\mathbb{E} X_1 = p$ and $\text{Var}\, X_1 = p(1-p)$.

(b) Let $g(z) = z(1-z)$. Then $g'(z) = z(-1) + 1(1-z) = 1-2z$.

We have $[g'(p)]^2 p(1-p) = (1-2p)^2 p(1-p)$, so the delta method gives

$$\sqrt{n}\,(Y_n(1-Y_n) - p(1-p)) \xrightarrow{D} N(0, (1-2p)^2 p(1-p)),$$

provided $p \neq 1/2$. If $p = 1/2$ then $g'(p) = 0$, so we cannot use the 1^{st}-order delta method.

(c) For $g(z) = z(1-z)$ we have $g''(z) = -2$.

If $p = 1/2$, $\dfrac{g''(p)\; p(1-p)}{2} = \dfrac{(-2)\frac{1}{2}\frac{1}{2}}{2} = -\frac{1}{4}$.

& the 2^{nd}-order delta method gives

$$n\left(Y_n(1-Y_n) - \frac{1}{4}\right) \xrightarrow{D} -\frac{1}{4} W, \quad \text{where} \quad W \sim \chi^2_1.$$

$\boxed{5.51}$ For $U_1, \ldots, U_n \overset{ind}{\sim} \text{Unif}(0,1)$,

(a)
$$\frac{\sqrt{n}\,(\bar{U}_n - \mathbb{E}U_1)}{\sqrt{\text{Var}\,U_1}} = \frac{\sqrt{n}\,(\bar{U}_n - 1/2)}{\sqrt{1/12}} \xrightarrow{d} N(0,1).$$

For $n = 12$, we have

$$\frac{\sqrt{n}\,(\bar{U}_n - 1/2)}{\sqrt{1/12}} = (12)(\bar{U}_n - 1/2) = \sum_{i=1}^{12} U_i - 6 .$$

(b) Support of $\sum_{i=1}^{12} U_i - 6$ is only $(-6, 6)$.