STAT 713 sp 2023 Lec 06 slides

Best unbiased estimators, Rao-blackwell theorem

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Estimator

An *estimator* is any function of the data intended as a guess of a parameter value.

Some quantities for measuring the quality of estimators:

Bias, standard error, mean squared error

For an estimator $\hat{\theta}$ of $\theta \in \Theta \subset \mathbb{R}$:

1. The *bias* is defined as $\text{Bias}\,\hat{\theta} = \mathbb{E}\hat{\theta} - \theta$.
2. The *standard error (SE)* is defined as $\text{SE}\,\hat{\theta} = \sqrt{\text{Var}\,\hat{\theta}}$.
3. The *mean squared error (MSE)* is defined as $\text{MSE}\,\hat{\theta} = \mathbb{E}(\hat{\theta} - \theta)^2$.

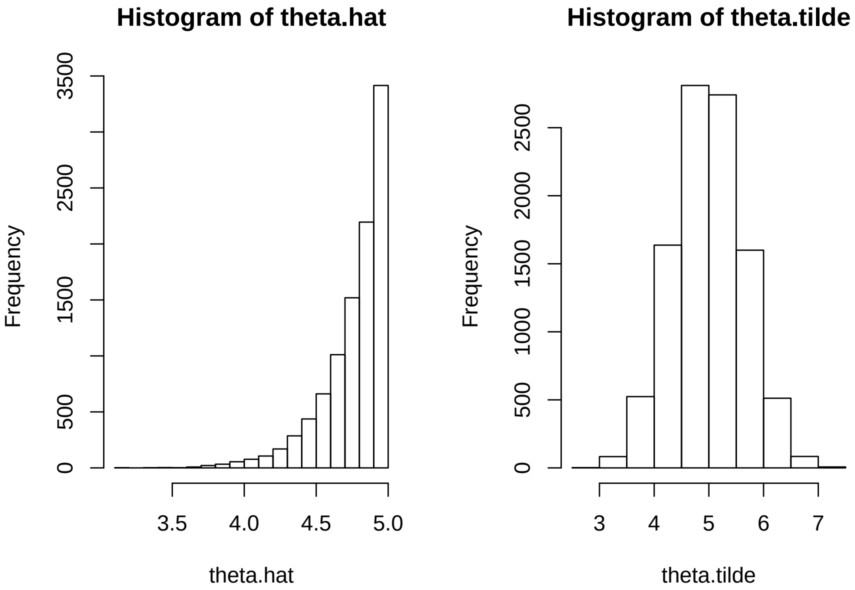
Exercise: Show that $\text{MSE}\,\hat{\theta} = \text{Var}\,\hat{\theta} + (\text{Bias}\,\hat{\theta})^2$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$ and consider two estimators of θ:

$$\hat{\theta} = X_{(n)}$$
$$\tilde{\theta} = 2\bar{X}_n$$

Which estimator is better?

1. Find $\text{Bias}\,\hat{\theta}$ and $\text{Bias}\,\tilde{\theta}$.
2. Find $\text{Var}\,\hat{\theta}$ and $\text{Var}\,\tilde{\theta}$.
3. Compare $\text{MSE}\,\hat{\theta}$ and $\text{MSE}\,\tilde{\theta}$ at different sample sizes $n = 1, 2, \dots$
4. Run a simulation to demonstrate $\text{MSE}\,\tilde{\theta}$ and $\text{MSE}\,\hat{\theta}$.
5. Propose a bias-corrected version $\hat{\theta}_{\text{unbiased}}$ of $\hat{\theta} = X_{(n)}$ and find $\text{MSE}\,\hat{\theta}_{\text{unbiased}}$.

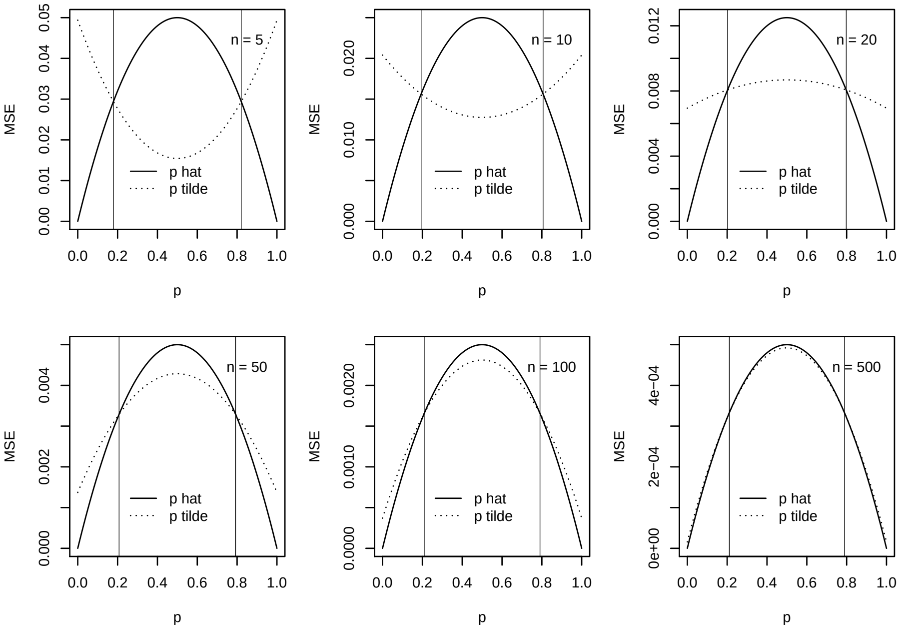

Histogram of theta.hat
Frequency
0
500
1500
2500
3500
3.5
4.0
4.5
5.0
theta.hat
Histogram of theta.tilde
Frequency
0
500
1000
1500
2000
2500
3
4
5
6
7
theta.tilde

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$ and consider two estimators of p:

$$\hat{p} = \frac{Y}{n}$$
$$\tilde{p} = \frac{Y+2}{n+4}, \qquad \text{where } Y = X_1 + \cdots + X_n.$$

Which estimator is better?

1. Find $\text{Bias}\,\hat{p}$ and $\text{Bias}\,\tilde{p}$.
2. Find $\text{Var}\,\hat{p}$ and $\text{Var}\,\tilde{p}$.
3. Compare $\text{MSE}\,\hat{p}$ and $\text{MSE}\,\tilde{p}$ at different values of p.



n = 5
n = 10
n = 20
n = 50
n = 100
n = 500
p hat
p tilde
MSE
p

Uniform minimum variance unbiased estimator (UMVUE)

An estimator $\hat{\theta}$ is a *UMVUE* for θ if it is unbiased and, for every other unbiased estimator $\tilde{\theta}$, $\operatorname{Var}\hat{\theta} \leq \operatorname{Var}\tilde{\theta}$ for all $\theta \in \Theta$.

Exercise: Which estimators are *not* UMVUEs in the following settings?

1. $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$, $\lambda > 0$, $\hat{\lambda}_1 = \bar{X}_n$, $\hat{\lambda}_2 = S_n^2$.
2. $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\theta)$, $\theta > 0$, $\hat{\theta}_1 = 2\bar{X}_n$, $\hat{\theta}_2 = X_{(n)}(n+1)/n$.
3. $X_1, \dots, x_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$, $p \in (0,1)$, $\hat{p}_1 = \bar{X}_n$, $\hat{p}_2 = \frac{\sum_{i=1}^n X_i + 2}{n+4}$.

Theorem (Rao-Blackwell, cf. Thm 7.3.17 in CB)

Let $\tilde{\tau}$ be an unbiased estimator for $\tau = \tau(\theta)$ and T be a sufficient statistic for θ. Then $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$ is an estimator that is unbiased for $\tau(\theta)$ with $\text{Var}\,\hat{\tau} \leq \text{Var}\,\tilde{\tau}$.

When we condition on a sufficient statistic, unbiased estimators

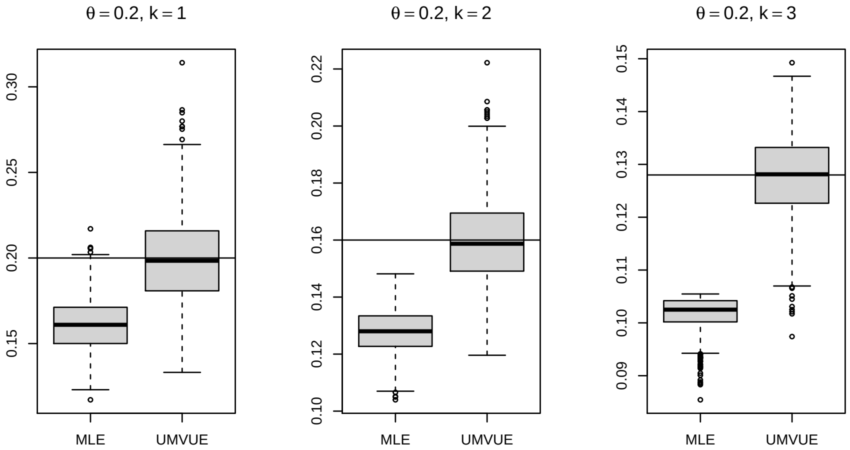
1. remain unbiased
2. may have smaller variance (but never greater)

So if you have $\tilde{\tau}$ which is unbiased, $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$ cannot be worse (might be better)!

Exercise: Prove the result.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Geometric}(\theta)$, $\theta \in (0, 1)$.

1. Find a silly unbiased estimator $\tilde{\tau}$ for $\tau = \tau(\theta) = \theta(1-\theta)^{k-1}$.
2. See if you can improve it by "Rao-Blackwell-ization".
3. Compare your Rao-Blackwell-ized estimator to the MLE for τ.



1000 estimates of $\theta(1-\theta)^{k-1}$ from $X_1,\ldots X_n$ ind Geom(θ) with $n = 50$

Theorem (Lehmann-Scheffé, cf. Thms 7.3.19 and 7.3.23 in CB)

Let T be a complete suff. stat. for θ and $\tilde{\tau}$ any unbiased estimator of $\tau = \tau(\theta)$. Then $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$ is the (unique) UMVUE for τ.

Means if $\mathbb{E}_\theta[h(T)] = \tau(\theta)$ for all $\theta \in \Theta$, then $h(T)$ is the UMVUE for $\tau(\theta)$.

Exercise: Go through proof.

Exercise: Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Poisson}(\theta)$, $\theta > 0$.

1. Find the UMVUE for $\tau = \tau(\theta) = P_\theta(X = 0) = e^{-\theta}$.
2. Compare this to the MLE for $\tau(\theta)$.

Steps to find the UMVUE (if it exists)

1. Find a complete sufficient statistic T.
2. Find a function $h(T)$ such that $\mathbb{E}_\theta h(T) = \tau(\theta)$ for all θ in one of these ways:
   - Identify the distribution of T and use this to determine h.
   - Start with any estimator $\tilde{\tau}$ that is unbiased for $\tau(\theta)$ and set $h(T) = \mathbb{E}[\tilde{\tau}|T]$.

Exercise: Find the UMVUEs for $\tau(\theta)$ in the following settings:

1. $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\theta) = \frac{1}{x\theta\sqrt{2\pi}} e^{-(\log x)^2/(2\theta^2)} \mathbf{1}(x > 0)$, $\theta > 0$, $\tau(\theta) = \theta^2$
2. $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(0, \theta)$, $\quad \theta > 0$, $\quad \tau(\theta) = \sqrt{\theta}$

Without completeness it is difficult (maybe impossible) to find a UMVUE.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\theta - 1/2, \theta + 1/2)$, for $\theta \in \mathbb{R}$.

1. Show that $T(X_1, \dots, X_n) = (X_{(1)}, X_{(n)})$ is minimal sufficient for θ.
2. For $M = (X_{(1)} + X_{(n)})/2$ and $R = X_{(n)} - X_{(1)}$, show

$$\begin{aligned} M|R &\sim \text{Uniform}(\theta - (1-R)/2, \theta + (1-R)/2) \\ R &\sim \text{Beta}(n-1, 2). \end{aligned}$$

3. Argue whether $T(X_1, \dots, X_n) = (X_{(1)}, X_{(n)})$ is complete.
4. Consider whether $\hat{\theta} = M$ is a UMVUE.
5. Check unbiasedness of $\tilde{\theta} = M + a(M+b)(R - (n-1)/(n+1))$, $a, b \in \mathbb{R}$.
6. At home: Compare $\text{Var}\,\hat{\theta}$ to $\text{Var}\,\tilde{\theta}$.

Theorem (UMVUE $\iff$ uncorrelated w/all unbiased estimators of 0)

Let $\hat{\tau}$ *be unbiased for* $\tau(\theta)$ *and let* $\mathbb{E}_\theta U = 0$ *and* $\mathbb{E}_\theta U^2 \in (0, \infty)$ *for all* θ.

1. *For a fixed* θ_0, *let* $a_0 = \dfrac{\text{Cov}_{\theta_0}(\hat{\tau}, U)}{\text{Var}_{\theta_0} U}$. *If* $a_0 \neq 0$ *then*
$$\text{Var}_{\theta_0}(\hat{\tau} - a_0 U) < \text{Var}_{\theta_0}(\hat{\tau}).$$
2. $\hat{\tau}$ *is the UMVUE iff it is uncorrelated with all unbiased estimators of zero.*

[cf. Thm 7.3.20 in CB]

Corollary (Sum of UMVUEs)

If $\hat{\tau}_1$ *and* $\hat{\tau}_2$ *are the UMVUEs for* τ_1 *and* τ_2, *respectively, then* $\hat{\tau}_1 + \hat{\tau}_2$ *is the UMVUE for* $\tau_1 + \tau_2$.

Exercise: Prove the results.