STAT 713 sp 2023 Lec 06 slides

Best unbiased estimators, Rao-blackwell theorem

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Estimator

An *estimator* is any function of the data intended as a guess of a parameter value.

Some quantities for measuring the quality of estimators:

Bias, standard error, mean squared error

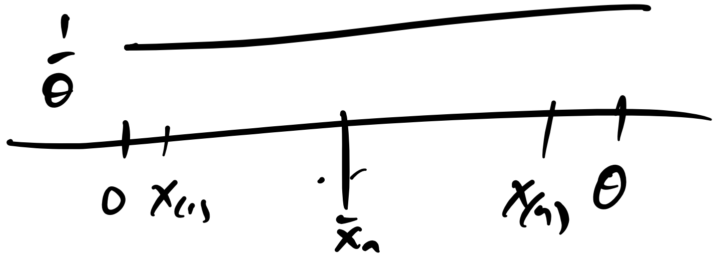
For an estimator $\hat{\theta}$ of $\theta \in \Theta \subset \mathbb{R}$:

1. The *bias* is defined as $\text{Bias}\,\hat{\theta} = \mathbb{E}\hat{\theta} - \theta$.
2. The *standard error (SE)* is defined as $\text{SE}\,\hat{\theta} = \sqrt{\text{Var}\,\hat{\theta}}$.
3. The *mean squared error (MSE)* is defined as $\text{MSE}\,\hat{\theta} = \mathbb{E}(\hat{\theta} - \theta)^2$.

Exercise: Show that $\text{MSE}\,\hat{\theta} = \text{Var}\,\hat{\theta} + (\text{Bias}\,\hat{\theta})^2$.

$$
\begin{aligned}
MSE\,\hat\theta &= \mathbb{E}\left(\hat\theta - \theta\right)^2 \\
&= \mathbb{E}\left(\left(\hat\theta - \mathbb{E}\hat\theta\right) + \left(\mathbb{E}\hat\theta - \theta\right)\right)^2 \\
&= \mathbb{E}\left(\left(\hat\theta - \mathbb{E}\hat\theta\right)^2 + 2\left(\hat\theta - \mathbb{E}\hat\theta\right)\left(\mathbb{E}\hat\theta - \theta\right) + \underline{\left(\mathbb{E}\hat\theta - \theta\right)^2}\right) \\
&= \mathbb{E}\left(\hat\theta - \mathbb{E}\hat\theta\right)^2 + \underbrace{2\,\overbrace{\mathbb{E}\left(\hat\theta - \mathbb{E}\hat\theta\right)}^{=0}\overset{\text{const.}}{\left(\mathbb{E}\hat\theta - \theta\right)}}_{=0} + \left(\mathbb{E}\hat\theta - \theta\right)^2 \\
&= \mathrm{Var}\,\hat\theta + \left(\mathrm{bias}\,\hat\theta\right)^2
\end{aligned}
$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(0, \theta)$ and consider two estimators of θ:



$\hat{\theta} = X_{(n)}$ $\quad \mathbb{E}\hat{\theta} = \frac{n}{n+1}\theta$ $\quad \text{Var}\,\hat{\theta} = \theta^2 \frac{n}{(n+2)(n+1)^2}$

$\tilde{\theta} = 2\bar{X}_n$ $\quad \mathbb{E}\tilde{\theta} = \theta$ $\quad \text{Var}\,\tilde{\theta} = \frac{\theta^2}{3n}$

Which estimator is better?

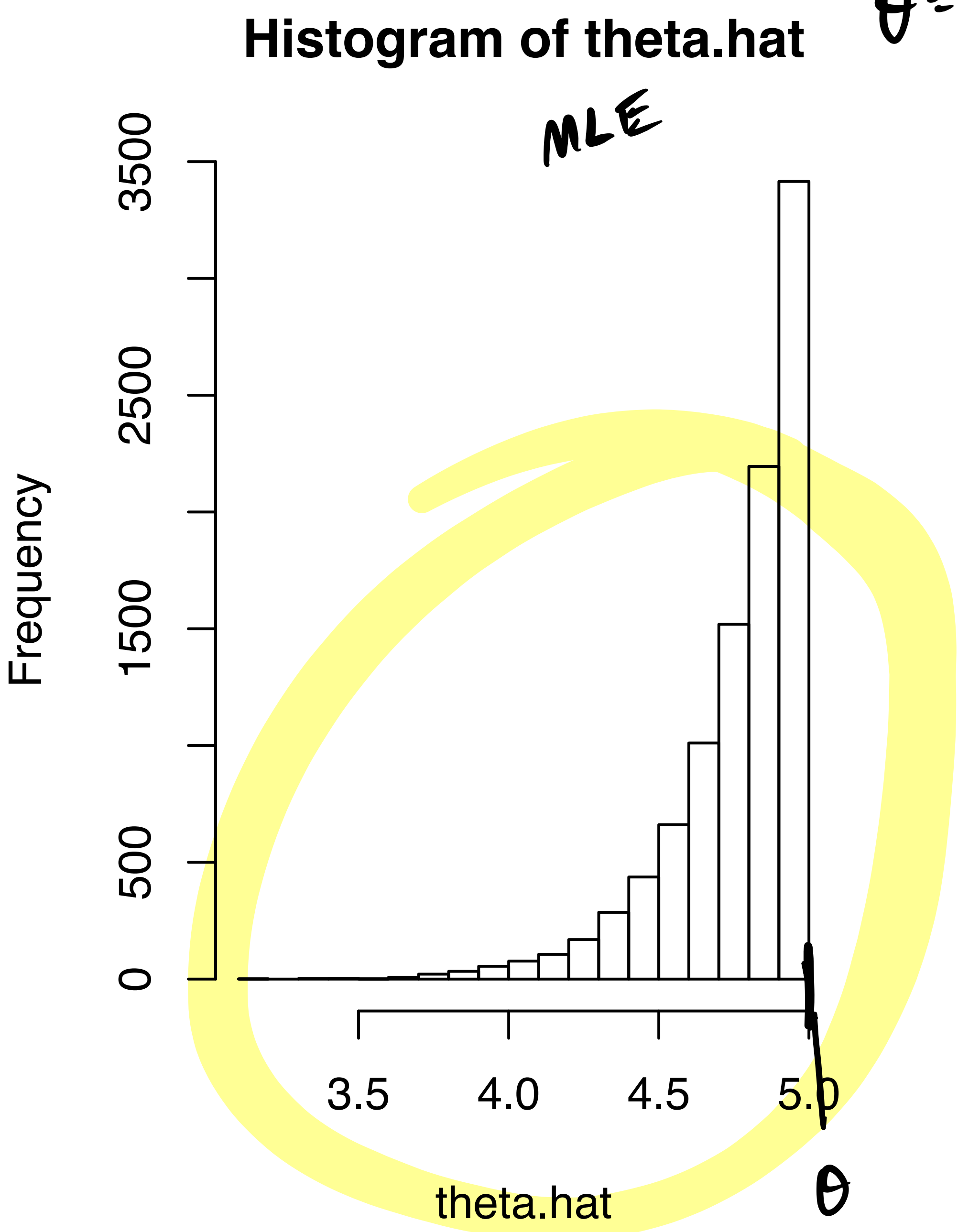
$$\text{MSE}\,\hat{\theta} = \theta^2 \frac{n}{(n+2)(n+1)^2} + \left[\frac{n}{n+1}\theta - \theta\right]^2$$

$$\text{MSE}\,\tilde{\theta} = \frac{\theta^2}{3n}$$

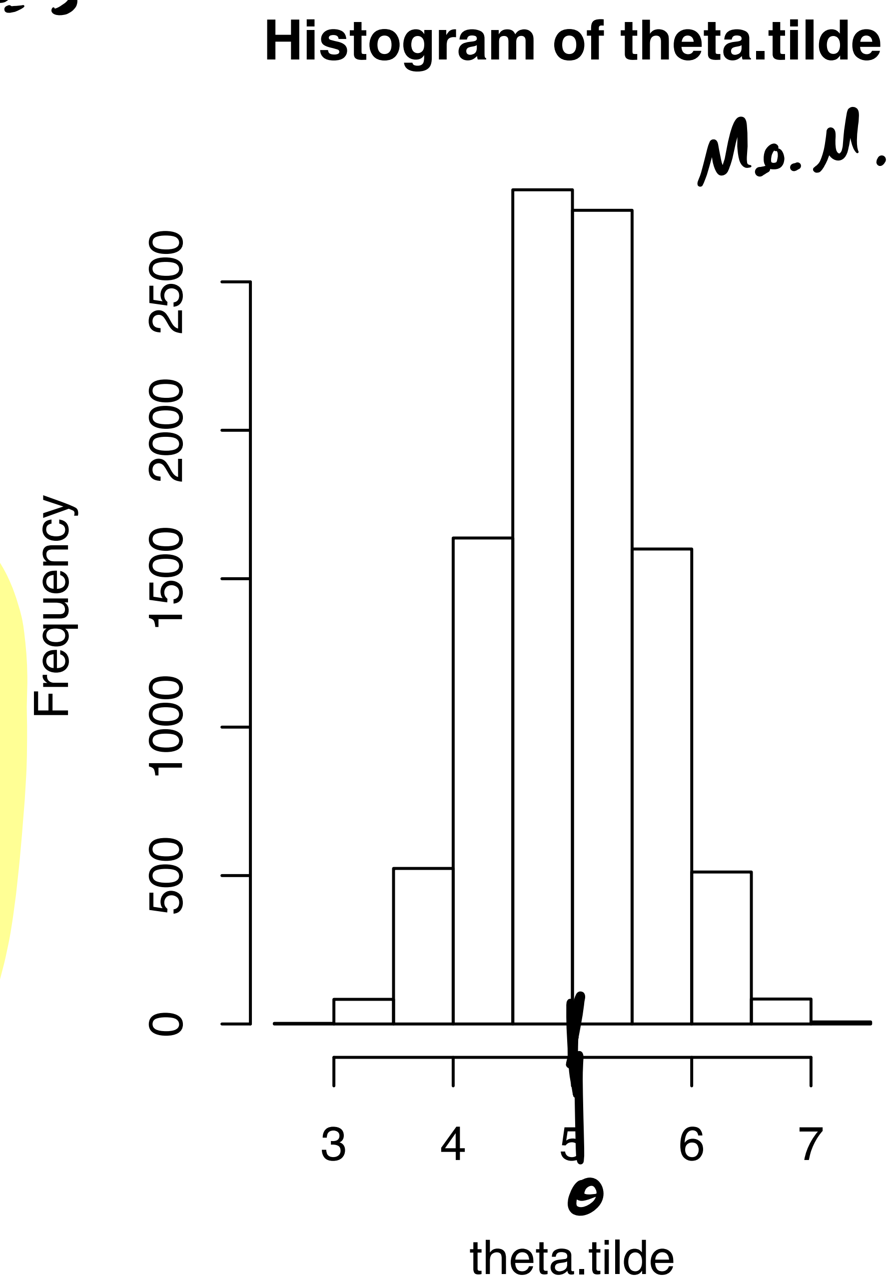
1. Find $\text{Bias}\,\hat{\theta}$ and $\text{Bias}\,\tilde{\theta}$.
2. Find $\text{Var}\,\hat{\theta}$ and $\text{Var}\,\tilde{\theta}$.
3. Compare $\text{MSE}\,\hat{\theta}$ and $\text{MSE}\,\tilde{\theta}$ at different sample sizes $n = 1, 2, \dots$
4. Run a simulation to demonstrate $\text{MSE}\,\tilde{\theta}$ and $\text{MSE}\,\hat{\theta}$.
5. Propose a bias-corrected version $\hat{\theta}_{\text{unbiased}}$ of $\hat{\theta} = X_{(n)}$ and find $\text{MSE}\,\hat{\theta}_{\text{unbiased}}$.

Let $\hat{\theta}_{\text{unbiased}} = \frac{n+1}{n} X_{(n)}$. Then $\mathbb{E}\hat{\theta}_{\text{unbiased}} = \frac{n+1}{n}\mathbb{E}X_{(n)} = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta$.

$\theta = 5$



Histogram of theta.hat
MLE
Frequency
0
500
1500
2500
3500
3.5
4.0
4.5
5.0
theta.hat
θ



Histogram of theta.tilde
Mo.M.
Frequency
0
500
1000
1500
2000
2500
3
4
5
6
7
θ
theta.tilde

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$ and consider two estimators of p:

$$\hat{p} = \frac{Y}{n} \quad \text{MLE}$$

$$\tilde{p} = \frac{Y+2}{n+4}, \qquad \text{where } Y = X_1 + \cdots + X_n.$$

$Y \mid p \sim \text{Binomial}(n, p)$

$p \sim \text{Beta}(\alpha = 2, \beta = 2)$

Then $\mathbb{E}[p \mid Y] = \dfrac{Y+2}{n+4}$.

Which estimator is better?

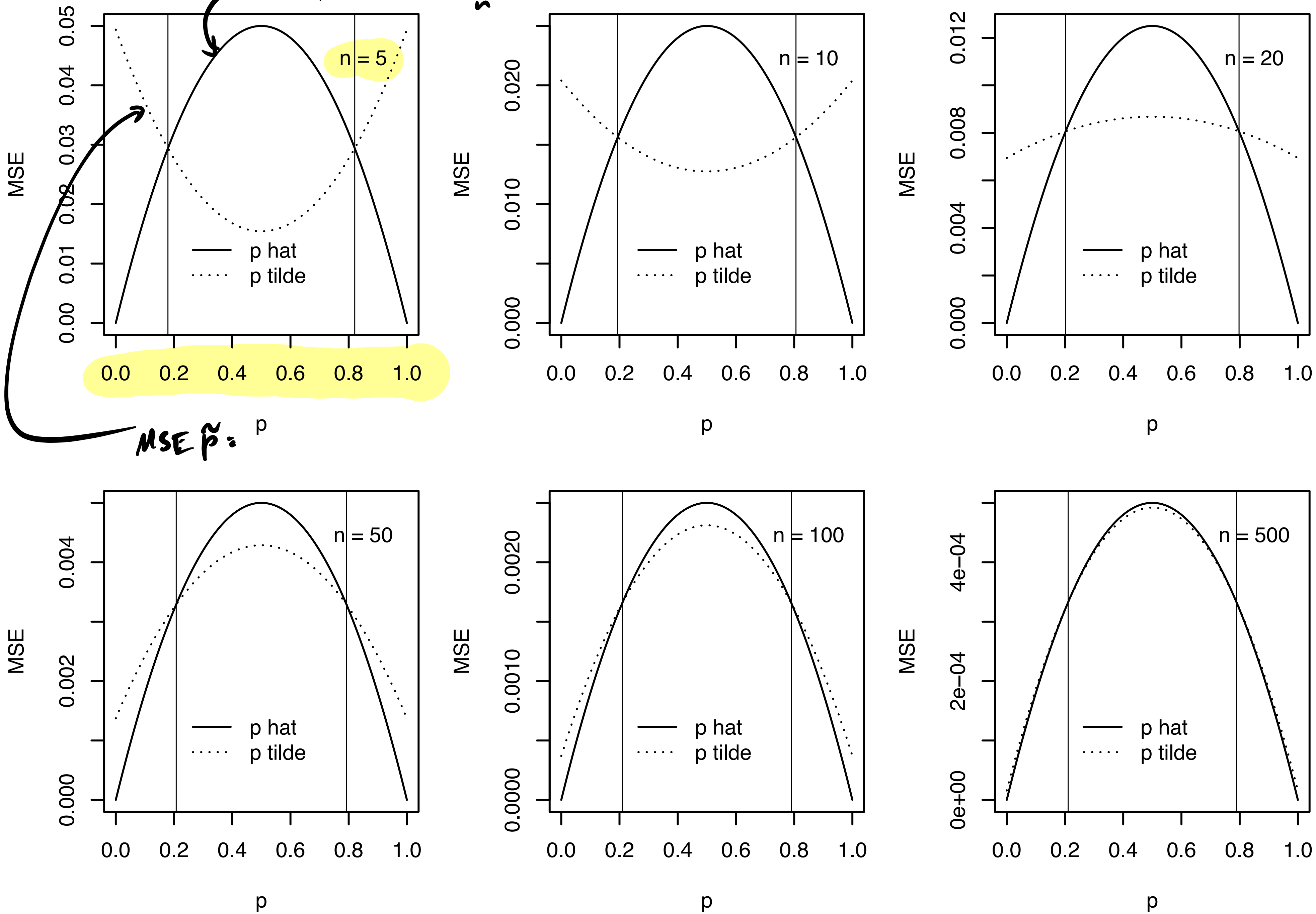
🤔

$\mathbb{E}\,\hat{p} = p \qquad \text{Var}\,\hat{p} = \dfrac{p(1-p)}{n}$

$\mathbb{E}\,\tilde{p} = p\left(\dfrac{n}{n+4}\right) + \dfrac{2}{n+4}, \qquad \text{Var}\,\tilde{p} = \dfrac{p(1-p)}{n}\left(\dfrac{n}{n+4}\right)^2$

1. Find Bias $\hat{p}$ and Bias $\tilde{p}$.
2. Find Var $\hat{p}$ and Var $\tilde{p}$.
3. Compare MSE $\hat{p}$ and MSE $\tilde{p}$ at different values of p.

MSE $\hat{p}$ = Var $\hat{p}$ = $\frac{p(1-p)}{n}$



MSE $\tilde{p}$ =

END OF EXAM I MATERIAL.

Focus now on unbiased estimators.

Uniform minimum variance unbiased estimator (UMVUE)

An estimator $\hat{\theta}$ is a *UMVUE* for θ if it is unbiased and, for every other unbiased estimator $\tilde{\theta}$, $\text{Var}\,\hat{\theta} \leq \text{Var}\,\tilde{\theta}$ $\boxed{\text{for all } \theta \in \Theta.}$

"Uniform"

Exercise: Which estimators are *not* UMVUEs in the following settings?

1. $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$, $\lambda > 0$, $\hat{\lambda}_1 = \bar{X}_n$, ? $\hat{\lambda}_2 = S_n^2$. ?
2. $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\theta)$, $\theta > 0$, $\cancel{\hat{\theta}_1 = 2\bar{X}_n}$, $\hat{\theta}_2 = X_{(n)}(n+1)/n$. ?
3. $X_1, \dots, x_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$, $p \in (0,1)$, $\hat{p}_1 = \bar{X}_n$, ? $\hat{p}_2 = \cancel{\frac{\sum_{i=1}^n X_i + 2}{n+4}}$.

Theorem (Rao-Blackwell, cf. Thm 7.3.17 in CB)

Let $\tilde{\tau}$ *be an unbiased estimator for* $\boxed{\tau = \tau(\theta)}$ *and* T *be a sufficient statistic for* θ*. Then* $\boxed{\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]}$ *is an estimator that is unbiased for* $\tau(\theta)$ *with* $\text{Var}\,\hat{\tau} \leq \text{Var}\,\tilde{\tau}$*.*

make new estimator

When we condition on a sufficient statistic, unbiased estimators

1. remain unbiased
2. may have smaller variance (but never greater)

So if you have $\tilde{\tau}$ which is unbiased, $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$ cannot be worse (might be better)!

Exercise: Prove the result.

Proof of RB:

Let $\mathbb{E}\,\tilde{\tau} = \tau = \tau(\theta)$, and T be suff. stat for θ

Set $\hat{\tau} = \mathbb{E}[\tilde{\tau} \mid T]$.

Want to show

(i) $\hat{\tau}$ is an estimator (a function we can compute on data)

(ii) $\mathbb{E}\,\hat{\tau} = \tau$

(iii) $\mathrm{Var}\,\hat{\tau} \leq \mathrm{Var}\,\tilde{\tau}$.

For (i), note that since T is suff, $\mathbb{E}[\tilde{\tau} \mid T]$ does not depend on θ.

(ii) $$\mathbb{E}\,\hat{\tau} = \mathbb{E}\left(\mathbb{E}[\tilde{\tau} \mid T]\right) = \mathbb{E}\,\tilde{\tau} = \tau$$

(iii) $$\mathrm{Var}\,\tilde{\tau} = \mathrm{Var}\big(\underbrace{\mathbb{E}[\tilde{\tau} \mid T]}_{\hat{\tau}}\big) + \mathbb{E}\left(\mathrm{Var}[\tilde{\tau} \mid T]\right)$$

$$= \mathrm{Var}\,\hat{\tau} + \underbrace{\mathbb{E}\left(\mathrm{Var}[\tilde{\tau} \mid T]\right)}_{\geq 0}$$

$$\Rightarrow \quad \mathrm{Var}\,\hat{\tau} \leq \mathrm{Var}\,\tilde{\tau}. \qquad \square$$

$$p_X(x;\theta) = \underset{\substack{\uparrow \\ \text{1 success}}}{\theta}\,\underbrace{(1-\theta)^{x-1}}_{x-1 \text{ failures}}$$

$$\tau = \tau(\theta) = \theta(1-\theta)^{k-1} = P(X=k)$$

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Geometric}(\theta)$, $\theta \in (0,1)$.

1. Find a silly unbiased estimator $\tilde{\tau}$ for $\tau = \tau(\theta) = \theta(1-\theta)^{k-1}$.
2. See if you can improve it by "Rao-Blackwell-ization".
3. Compare your Rao-Blackwell-ized estimator to the MLE for τ.

① Let $\tilde{\tau} = \mathbb{1}(X_1 = k)$.

Then
$$\begin{aligned}
\mathbb{E}\tilde{\tau} &= \mathbb{E}\,\mathbb{1}(X_1=k) \\
&= 0 \cdot P(\mathbb{1}(X_1=k)=0) + 1 \cdot P(\mathbb{1}(X_1=k)=1) \\
&= P(X_1=k) \\
&= \theta(1-\theta)^{k-1} \\
&= \tau.
\end{aligned}$$

② Find a suff stat for θ.

$$p(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i - n}$$

So $T = \sum_{i=1}^{n} X_i$ is suff for θ.

R.B.:

Set $\hat{\tau} = \mathbb{E}[\tilde{\tau} \mid T]$.

Will be nicer to write

$$\hat{\tau}(t) = \mathbb{E}[\tilde{\tau} \mid T = t]$$

$$= \mathbb{E}\left[\mathbb{1}(X_1 = k) \,\middle|\, \sum_{i=1}^{n} X_i = t\right]$$

$$= P\left(X_1 = k \,\middle|\, \sum_{i=1}^{n} X_i = t\right)$$

$T_n = \sum_{i=1}^{n} X_i$

$$= \frac{P\left(X_1 = k \cap \sum_{i=1}^{n} X_i = t\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$T \sim p_T(t) = \binom{t-1}{n-1}\theta^n (1-\theta)^{t-n}$

$$= \frac{P(X_1 = k)\cdot P\Big(\underbrace{\sum_{i=2}^{n} X_i}_{n-1 \text{ success}} = t-k\Big)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$t-k \geq n-1$

$t \geq n+k-1$

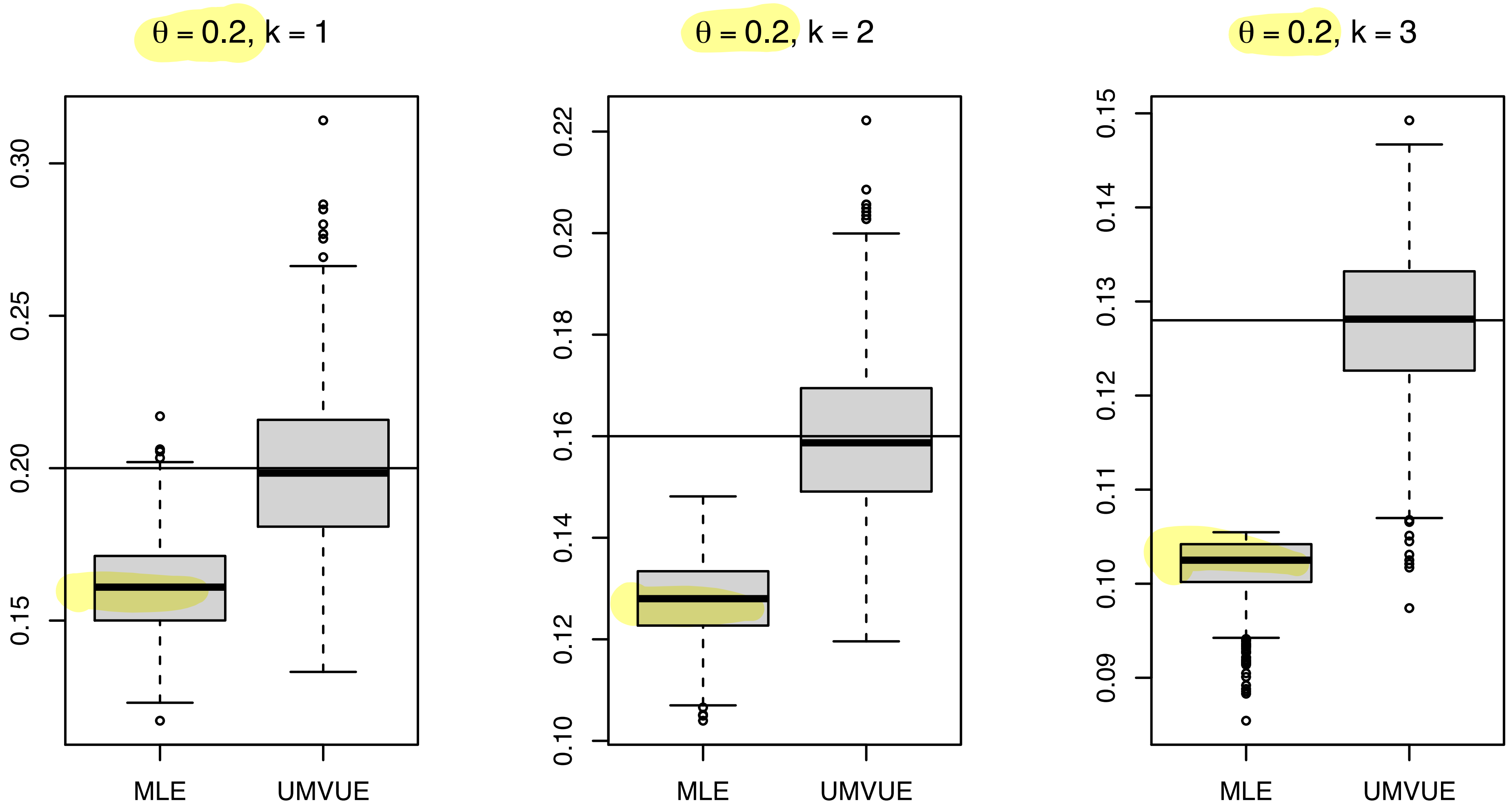
$$= \frac{\theta(1-\theta)^{k-1}\binom{t-k-1}{n-2}\theta^{n-1}(1-\theta)^{t-k-(n-1)}}{\binom{t-1}{n-1}\theta^n (1-\theta)^{t-n}}$$

$$= \frac{\cancel{\theta}\,\cancel{(1-\theta)^{k-1}} \binom{t-k-1}{n-2} \cancel{\theta^{n-1}}\,\cancel{(1-\theta)^{t-k-(n-1)}}}{\binom{t-1}{n-1} \cancel{\theta^{n}}\,\cancel{(1-\theta)^{t-n}}}$$

$$= \frac{\binom{t-k-1}{n-2}}{\binom{t-1}{n-1}} \qquad \text{for} \quad t \geq n+k-1 .$$

$$\hat{\tau} = \frac{\binom{\sum_{i=1}^{n} x_i - k - 1}{n-2}}{\binom{\sum_{i=1}^{n} x_i - 1}{n-1}}$$

$\tau(\theta) = \theta(1-\theta)^{k-1}$



1000 estimates of $\theta(1-\theta)^{k-1}$ from $X_1, \ldots X_n$ ind Geom(θ) with $\boxed{n = 50}$

Theorem (Lehmann-Scheffé, cf. Thms 7.3.19 and 7.3.23 in CB)

Let T be a complete suff. stat. for θ and $\tilde{\tau}$ any unbiased estimator of $\tau = \tau(\theta)$. Then $\boxed{\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]}$ is the (unique) UMVUE for τ.

Means if $\mathbb{E}_\theta[h(T)] = \tau(\theta)$ for all $\theta \in \Theta$, then $h(T)$ is the UMVUE for $\tau(\theta)$.

Exercise: Go through proof.

The estimator $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$ is a function of T.

Also it is unbiased, so $\mathbb{E}\hat{\tau} = \tau$.

Suppose $h_1(T)$ and $h_2(T)$ are both functions of T such that $\mathbb{E}h_1(T) = \tau$ and $\mathbb{E}h_2(T) = \tau$. Two unbiased estimators for τ which are functions of T.

We have

$$\mathbb{E}\left[h_1(T) - h_2(T)\right] = 0 \quad \forall\ \theta$$

Since T is complete, this implies $h_1(T) - h_2(T) = 0$ w.p. 1.

This means $h_1(T)$ and $h_2(T)$ are the same.

So there exists only $\underline{\underline{\text{one}}}$ unbiased estimator of τ which is a function of T.

Now begin with any unbiased estimator $\tilde{\tau}$, we have

$$\hat{\tau} = \mathbb{E}\left[\tilde{\tau} \mid T\right] \quad \text{has} \quad \mathrm{Var}\ \hat{\tau} \le \mathrm{Var}\ \tilde{\tau}.$$

So $\hat{\tau}$ is the unique UMVUE.

$$\tau = \tau(\theta) = e^{-\theta} = P_\theta(X_1 = 0).$$

$$p_X(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\theta)$, $\theta > 0$.

$T = \sum X_i$ is complete suff. (by exponential fam).

1. Find the UMVUE for $\tau = \tau(\theta) = P_\theta(X = 0) = e^{-\theta}$.
2. Compare this to the MLE for $\tau(\theta)$.

① $\tilde{\tau} = \mathbb{1}(X_1 = 0)$, then $\mathbb{E}\,\tilde{\tau} = P(X_1 = 0) = e^{-\theta} = \tau.$

Do RB:

$$\hat{\tau}(t) = \mathbb{E}\left[\tilde{\tau} \,\middle|\, T = t\right]$$

$$= \mathbb{E}\left[\mathbb{1}(X_1 = 0) \,\middle|\, \sum_{i=1}^n X_i = t\right]$$

$$= P\left(X_1 = 0 \;\middle|\; \sum_{i=1}^{n} X_i = t \right)$$

$$= \frac{P\left(X_1 = 0 \cap \sum_{i=1}^{n} X_i = t \right)}{P\left(\sum_{i=1}^{n} X_i = t \right)}$$

$$= \frac{P\left(X_1 = 0 \right) P\left(\sum_{i=2}^{n} X_i = t \right)}{P\left(\sum_{i=1}^{n} X_i = t \right)}$$

$\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$

$\sum_{i=2}^{n} X_i \sim \text{Poisson}((n-1)\theta)$

$$= \frac{e^{-\theta} \, e^{-(n-1)\theta} \left((n-1)\theta\right)^t / t!}{\dfrac{e^{-n\theta} (n\theta)^t}{t!}}$$

$$= \left(\frac{n-1}{n} \right)^t$$

$$= \left(1 - \frac{1}{n} \right)^t .$$

So

$$\hat{\tau} = \left(1 - \frac{1}{n} \right)^{\sum_{i=1}^{n} X_i} = \Big[\underbrace{\left(1 - \frac{1}{n} \right)^n}_{\to e^{-1}} \Big]^{\bar{X}_n} \approx e^{-\bar{X}_n} \quad \text{for large } n.$$

$$\hat{\tau}_{mle} = \tau(\bar{X}_n) = e^{-\bar{X}_n}$$

Steps to find the UMVUE (if it exists)

1. Find a complete sufficient statistic T.
2. Find a function $h(T)$ such that $\mathbb{E}_\theta h(T) = \tau(\theta)$ for all θ in one of these ways:
   - Identify the distribution of T and use this to determine h.
   - Start with any estimator $\tilde{\tau}$ that is unbiased for $\tau(\theta)$ and set $h(T) = \mathbb{E}[\tilde{\tau}|T]$.

Exercise: Find the UMVUEs for $\tau(\theta)$ in the following settings:

1. $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f(x;\theta) = \frac{1}{x\theta\sqrt{2\pi}} e^{-(\log x)^2/(2\theta^2)} \mathbf{1}(x > 0)$, $\theta > 0$, $\tau(\theta) = \theta^2$ ✓
2. $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Normal}(0, \theta)$, $\quad \theta > 0, \quad \tau(\theta) = \sqrt{\theta}$

① $X_1, \dots, X_n \sim f(x;\theta) = \frac{1}{x\theta\sqrt{2\pi}} e^{-(\log x)^2/(2\theta^2)} \mathbb{1}(x>0)$

Find UMVUE for $\tau = \tau(\theta) = \theta^2$.

$t_1(x) = (\log x)^2 \qquad w_1(\theta) = -\frac{1}{2\theta^2}$

$T = \sum_{i=1}^n (\log X_i)^2$ is complete suff.

$f(x;\theta) = h(x)\, c(\theta) \exp\left[t_1(x) w_1(\theta) + \dots + t_k(x) w_k(\theta) \right]$

Set $Y = \log X$, $\quad y = \log x = g(x) \quad x = e^y = g^{-1}(y)$, $\quad \frac{d}{dy} g^{-1}(y) = e^y$

$\mathcal{Y} = \mathbb{R}$

$f_Y(y;\theta) = \frac{1}{e^y \theta \sqrt{2\pi}} e^{-y^2/(2\theta^2)} \left| e^y \right| = \frac{1}{\sqrt{2\pi}\,\theta} e^{-y^2/2\theta^2}$

$T = \sum_{i=1}^n Y_i^2 \quad , \quad Y_i = \log X_i \sim N(0, \theta^2)$

$\mathbb{E}\, T = \sum_{i=1}^n \mathbb{E}\, Y_i^2 = n\theta^2$.

$$\boxed{\hat{\theta}_{UMVUE} = \frac{T}{n} = \frac{1}{n} \sum_{i=1}^n (\log X_i)^2}$$

② $X_1, \dots X_n \overset{iid}{\sim} \text{Normal}(0, \underset{\text{variance}}{\theta})$, $\quad \tau = \tau(\theta) = \underset{\text{std. dev.}}{\sqrt{\theta}}$.

Find UMVUE for $\sqrt{\theta}$.

$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{\theta}}$, $\qquad T = \sum_{i=1}^n X_i^2$ is complete suff. stat.

We have $\mathbb{E}\sum_{i=1}^{n} X_i^2 = n\theta$.

$$Y = \sum_{i=1}^{n}\left(\frac{X_i}{\sqrt{\theta}}\right)^2 \sim \chi^2_n$$

$$\frac{X_i}{\sqrt{\theta}} \sim N(0,1)$$

$$\mathbb{E}\sqrt{Y} = \mathbb{E}\sqrt{\sum_{i=1}^{n}\left(\frac{X_i}{\sqrt{\theta}}\right)^2} = \frac{1}{\sqrt{\theta}}\,\mathbb{E}\sqrt{\sum_{i=1}^{n} X_i^2} =$$

$$Y \sim \chi^2_n, \quad f_Y(y) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}}\, y^{n/2-1} e^{-y/2}\, \mathbb{1}(y>0)$$

$$\text{Gamma}\left(\tfrac{n}{2}, 2\right)$$

$$\mathbb{E}\sqrt{Y} = \int_0^\infty \sqrt{y}\,\frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}}\, y^{n/2-1} e^{-y/2}\, dy$$

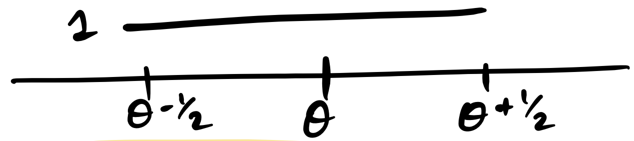
$$= \frac{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \underbrace{\int_0^\infty \frac{1}{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}}\, y^{\frac{n+1}{2}-1} e^{-y/2}\, dy}_{=1}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mathbb{E}\sqrt{Y} = \mathbb{E}\sqrt{\sum_{i=1}^{n}\left(\frac{X_i}{\sqrt{\theta}}\right)^2} = \frac{1}{\sqrt{\theta}}\,\mathbb{E}\sqrt{\sum_{i=1}^{n} X_i^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\Rightarrow \mathbb{E}\left(\sqrt{\sum_{i=1}^{n} X_i^2}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)}\right) = \sqrt{\theta}.$$

UMVUE because it is a function of a complete suff. statistic.



Without completeness it is difficult (maybe impossible) to find a UMVUE.

$f(x;\theta) = \mathbb{1}(\theta - ½ < x < \theta + ½)$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\theta - 1/2, \theta + 1/2)$, for $\theta \in \mathbb{R}$.

1. Show that $T(X_1, \dots, X_n) = (X_{(1)}, X_{(n)})$ is minimal sufficient for θ.
2. For $M = (X_{(1)} + X_{(n)})/2$ and $R = X_{(n)} - X_{(1)}$, show

$$M|R \sim \text{Uniform}(\theta - (1-R)/2, \theta + (1-R)/2)$$
$$R \sim \text{Beta}(n-1, 2).$$

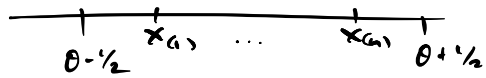
3. Argue whether $T(X_1, \dots, X_n) = (X_{(1)}, X_{(n)})$ is complete. NOT COMPLETE.
4. Consider whether $\hat{\theta} = M$ is a UMVUE.
5. Check unbiasedness of $\tilde{\theta} = M + a(M+b)(R - (n-1)/(n+1))$, $a, b \in \mathbb{R}$.
6. At home: Compare $\text{Var}\,\hat{\theta}$ to $\text{Var}\,\tilde{\theta}$.

↑ For any θ, we can find constants $a, b \in \mathbb{R}$ such that $\text{Var}\,\tilde{\theta} \le \text{Var}\,\hat{\theta}$.

$X_1,\dots,X_n \overset{iid}{\sim} f(x;\theta) = \mathbb{1}(\theta - 1/2 < x < \theta + 1/2)$

check if $T = (X_{(1)}, X_{(n)})$ is a min suff stat.

$$f(\underline{x};\theta) = \prod_{i=1}^{n} \mathbb{1}(\theta - 1/2 < x_i < \theta + 1/2) = \mathbb{1}(x_{(1)} > \theta - 1/2)\,\mathbb{1}(x_{(n)} < \theta + 1/2)$$



For two samples $\underline{x}, \underline{y}$,

$$\frac{f(\underline{x};\theta)}{f(\underline{y};\theta)} = \frac{\mathbb{1}(x_{(1)} > \theta - 1/2)\ \mathbb{1}(x_{(n)} < \theta + 1/2)}{\mathbb{1}(y_{(1)} > \theta - 1/2)\ \mathbb{1}(y_{(n)} < \theta + 1/2)}$$

num and denom positive (and constant) iff $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$.

Unif(a,b) has variance $\frac{(b-a)^2}{12}$

❷ For $M = (X_{(1)} + X_{(n)})/2$ and $R = X_{(n)} - X_{(1)}$, show

$$M|R \sim \text{Uniform}(\theta - (1-R)/2, \theta + (1-R)/2)$$
$$R \sim \text{Beta}(n-1, 2).$$

Beta(α,β) has mean $\frac{\alpha}{\alpha+\beta}$

var $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$\mathbb{E}\,M = \mathbb{E}\left(\mathbb{E}[M|R]\right) = \mathbb{E}(\theta) = \theta.$$

$$\begin{aligned}
\text{Var}\,M &= \text{Var}\left(\mathbb{E}[M|R]\right) + \mathbb{E}\left(\text{Var}[M|R]\right) \\
&= \text{Var}(\theta) + \mathbb{E}\left(\frac{(1-R)^2}{12}\right) \\
&= 0 + \frac{1}{12}\mathbb{E}\left[1 - 2R + R^2\right] \\
&= \frac{1}{12}\left[1 - 2\,\mathbb{E}R + \text{Var}\,R + (\mathbb{E}R)^2\right] \\
&= \frac{1}{12}\left[1 - 2\frac{(n-1)}{n-1+2} + \frac{(n-1)2}{(n-1+2)^2(n-1+2+1)} + \left(\frac{n-1}{n-1+2}\right)^2\right]
\end{aligned}$$

Theorem (UMVUE $\iff$ uncorrelated w/all unbiased estimators of 0)

Let $\hat{\tau}$ be unbiased for $\tau(\theta)$ and let $\mathbb{E}_\theta U = 0$ and $\mathbb{E}_\theta U^2 \in (0, \infty)$ for all θ.

1. *For a fixed θ_0, let* $a_0 = \dfrac{\mathrm{Cov}_{\theta_0}(\hat{\tau}, U)}{\mathrm{Var}_{\theta_0} U}$. *If $a_0 \neq 0$ then*

$$\mathrm{Var}_{\theta_0}(\hat{\tau} - a_0 U) < \mathrm{Var}_{\theta_0}(\hat{\tau}).$$

any rv with expected value 0.

2. *$\hat{\tau}$ is the UMVUE iff it is uncorrelated with all unbiased estimators of zero.*

[cf. Thm 7.3.20 in CB]

Corollary (Sum of UMVUEs)

If $\hat{\tau}_1$ and $\hat{\tau}_2$ are the UMVUEs for τ_1 and τ_2, respectively, then $\hat{\tau}_1 + \hat{\tau}_2$ is the UMVUE for $\tau_1 + \tau_2$.

Exercise: Prove the results.

Let $\hat{\tau}$ unbiased for τ, U a r.v. with $\mathbb{E}\, U = 0$
$\text{Var}\, U \in (0, \infty)$

①

$$\text{Var}(\hat{\tau} - a_0 U) = \text{Var}(\hat{\tau}) + a_0^2 \,\text{Var}\, U - 2 a_0 \,\text{Cov}(\hat{\tau}, U).$$

If $a_0 = \dfrac{\text{Cov}(\hat{\tau}, U)}{\text{Var}\, U} \neq 0$, we have

$$\text{Var}(\hat{\tau} - a_0 U) = \text{Var}\,\hat{\tau} + \left[\frac{\text{Cov}(\hat{\tau}, U)}{\text{Var}\, U}\right]^2 \text{Var}\, U - 2\,\frac{[\text{Cov}(\hat{\tau}, U)]^2}{\text{Var}\, U}$$

$$= \text{Var}\,\hat{\tau} - \underbrace{\frac{[\text{Cov}(\hat{\tau}, U)]^2}{\text{Var}\, U}}_{> 0}$$

$$\Rightarrow \quad \text{Var}(\hat{\tau} - a_0 U) < \text{Var}\,\hat{\tau}$$

So I can improve an unbiased estimator $\hat{\tau}$ if it is correlated with some r.v. with expectation 0.

② $\hat{\tau}$ is UMVUE $\iff$ $\text{Corr}(\hat{\tau}, U) = 0$ $\forall\, U$ s.t. $\mathbb{E}\, U = 0$.

"$\Rightarrow$" Let $\hat{\tau}$ be the UMVUE. Suppose $\text{Corr}(\hat{\tau}, U) \neq 0$.

Then $a_0 = \dfrac{\text{Cov}(\hat{\tau}, U)}{\text{Var}\, U} \neq 0$.

If so, then

$$\text{Var}(\hat{\tau} - a_0 U) < \text{Var}\,\hat{\tau}.$$

But this is a contradiction because $\hat{\tau}$ is the UMVUE.

So we must have $Corr(\hat{\tau}, U) = 0$.

"$\Leftarrow$" Let $Corr(\hat{\tau}, U) = 0 \quad \forall \; U$ s.t. $\mathbb{E}\, U = 0$.

Introduce any unbiased estimator $\tilde{\tau}$.

Then $\tilde{\tau} = \hat{\tau} + (\tilde{\tau} - \hat{\tau})$.

$$\text{Then} \quad Var\, \tilde{\tau} = Var\, \hat{\tau} + Var(\tilde{\tau} - \hat{\tau}) + \overbrace{2\, Cov(\hat{\tau}, \underbrace{\tilde{\tau} - \hat{\tau}}_{U})}^{=0}$$

$\mathbb{E}[\tilde{\tau} - \hat{\tau}] = 0$

$$= Var\, \hat{\tau} + \underbrace{Var(\tilde{\tau} - \hat{\tau})}_{\geq 0}$$

$$\Rightarrow \quad Var\, \tilde{\tau} \geq Var\, \hat{\tau}.$$

So $\hat{\tau}$ is the UMVUE.

HW:

$$x_1, \ldots x_n \sim f(x; \alpha, \beta) = \beta \alpha^{\beta} x^{-(\beta+1)} \mathbb{1}(x > \alpha)$$

$$F(x; \alpha, \beta) = \int_{\alpha}^{x} \beta \alpha^{\beta} t^{-(\beta+1)} \, dt$$

$$= \beta \alpha^{\beta} \left[\frac{t^{-\beta}}{-\beta} \right] \Bigg|_{\alpha}^{x}$$

$$= \beta \alpha^{\beta} \left[\frac{x^{-\beta}}{-\beta} + \frac{\alpha^{-\beta}}{\beta} \right]$$

$$= 1 - \left(\frac{x}{\alpha}\right)^{-\beta}.$$

$$u = 1 - \left(\frac{x}{\alpha}\right)^{-\beta} \iff 1 - u = \left(\frac{x}{\alpha}\right)^{-\beta}$$

$$\alpha (1-u)^{-\frac{1}{\beta}} = x$$

$u = 1/2$, $\quad q_{1/2} = \alpha(1 - 1/2)^{-1/\beta}$

$u = 1/8$ $\quad q_{1/8} = \alpha(1 - 1/8)^{-1/\beta}$

$X_{(\lceil \frac{1}{2} n \rceil)}$