$\boxed{\text{Theorem: Properties of Score and Fisher Information AND Cramér-Rao Lower Bound.}}$

The CR-condition is

$$\frac{\partial}{\partial\theta} \mathbb{E}_\theta h(\underset{\sim}{X}) = \mathbb{E}_\theta\left(h(\underset{\sim}{X}) \frac{\partial}{\partial\theta} \log f(\underset{\sim}{X};\theta)\right) \quad \text{whenever} \quad \mathbb{E}_\theta|h(\underset{\sim}{X})| < \infty$$

for all θ. In consequence of this condition, we have:

① $\mathbb{E}_\theta\, S(\theta;\underset{\sim}{X}) = 0$

__PROOF:__ Take $h(\underset{\sim}{X}) = 1$. Then $\frac{\partial}{\partial\theta} \mathbb{E}_\theta h(\underset{\sim}{X}) = 0$.

② $\mathbb{E}_\theta|h(\underset{\sim}{X})| < \infty \implies \frac{\partial}{\partial\theta} \mathbb{E}_\theta h(\underset{\sim}{X}) = \mathrm{Cov}_\theta(h(\underset{\sim}{X}), S(\theta;\underset{\sim}{X}))$

__PROOF:__ Since $\mathbb{E}_\theta\, S(\theta;\underset{\sim}{X}) = 0$, $\mathbb{E}_\theta\, h(\underset{\sim}{X}) S(\theta;\underset{\sim}{X}) = \mathrm{Cov}_\theta(h(\underset{\sim}{X}), S(\theta;\underset{\sim}{X}))$.

③ $I(\theta) = \mathrm{Var}_\theta\, S(\theta;\underset{\sim}{X})$

__PROOF:__ Since $\mathbb{E}_\theta S(\theta;\underset{\sim}{X}) = 0$, $\mathrm{Var}_\theta\, S(\theta;\underset{\sim}{X}) = \mathbb{E}\, S^2(\theta;\underset{\sim}{X})$.

④ If $\frac{\partial^2}{\partial\theta^2} f(\underset{\sim}{x};\theta)$ exists and $\frac{\partial}{\partial\theta} \mathbb{E}_\theta S(\theta;\underset{\sim}{X}) = \int_{\mathcal{X}} \frac{\partial}{\partial\theta} S(\theta;\underset{\sim}{x}) f(\underset{\sim}{x};\theta)\, d\underset{\sim}{x}$

$$I(\theta) = -\mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{X};\theta)\right)$$

__PROOF:__ Noting that $\frac{\partial}{\partial\theta} \mathbb{E}_\theta S(\theta;\underset{\sim}{X}) = 0$, we have

$$\begin{aligned} 0 &= \int_{\mathcal{X}} \frac{\partial}{\partial\theta} S(\theta;\underset{\sim}{x}) f(\underset{\sim}{x};\theta)\, d\underset{\sim}{x} \\ &= \int_{\mathcal{X}} \left[\frac{\partial}{\partial\theta} S(\theta;\underset{\sim}{x})\right] f(\underset{\sim}{x};\theta)\, d\underset{\sim}{x} + \int_{\mathcal{X}} S(\theta;\underset{\sim}{x}) \frac{\partial}{\partial\theta} f(\underset{\sim}{x};\theta)\, d\underset{\sim}{x} \end{aligned}$$

$$= \mathbb{E}_\theta \frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta) + \int_{\mathcal{X}} S(\theta;\underset{\sim}{x}) \underbrace{\frac{\frac{\partial}{\partial\theta} f(\underset{\sim}{x};\theta)}{f(\underset{\sim}{x};\theta)}}_{S(\theta;\underset{\sim}{x})} f(\underset{\sim}{x};\theta)\, d\underset{\sim}{x}$$

$$= \mathbb{E}_\theta \frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta) + I(\theta),$$

which gives the result.

CRLB: Let $\tilde{\tau}(\underset{\sim}{X})$ be an estimator with finite mean and suppose $\mathbb{E}_\theta \tilde{\tau}(\underset{\sim}{X})$ is differentiable w.r.t. θ. Then

$$\operatorname{Var} \tilde{\tau}(\underset{\sim}{X}) \geq \frac{\left[\frac{\partial}{\partial\theta} \mathbb{E}_\theta \tilde{\tau}(\underset{\sim}{X})\right]^2}{I(\theta)}.$$

PROOF: By the Cauchy-Schwarz inequality, we have

$$\operatorname{Cov}\left(\tilde{\tau}(\underset{\sim}{X}), S(\theta;\underset{\sim}{X})\right) \leq \sqrt{\operatorname{Var} \tilde{\tau}(\underset{\sim}{X})} \sqrt{\operatorname{Var} S(\theta;\underset{\sim}{X})}.$$

The right hand side is equal to $\frac{\partial}{\partial\theta} \mathbb{E}_\theta \tilde{\tau}(\underset{\sim}{X})$. Substituting this, squaring both sides, and rearranging gives the result.

$\boxed{\underset{\sim}{X} \overset{ind}{\sim} \text{Geometric}(\theta) \qquad \text{CRLB stuff}}$

① Find $\quad I(\theta) = \mathbb{E}\, S^2(\theta; \underset{\sim}{X})$.

We have $\quad f(\underset{\sim}{x};\theta) = \theta^n (1-\theta)^{n\bar{x}_n - n}$,

$$\log f(\underset{\sim}{x};\theta) = n \log \theta + (n\bar{x}_n - n) \log (1-\theta)$$

$$S(\theta; \underset{\sim}{X}) = \frac{\partial}{\partial\theta} \log f(\underset{\sim}{x};\theta) = \frac{n}{\theta} - \frac{n\bar{x} - n}{1-\theta}.$$

Then

$$I(\theta) = \text{Var}\, S(\theta; \underset{\sim}{X}) = \left(\frac{n}{1-\theta}\right)^2 \text{Var}\, \bar{X}_n = \left(\frac{n}{1-\theta}\right)^2 \frac{1}{n} \frac{1-\theta}{\theta^2} = \frac{n}{\theta^2(1-\theta)}.$$

② We have

$$\frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta) = \frac{\partial}{\partial\theta} S(\theta; \underset{\sim}{x}) = -\frac{n}{\theta^2} - \frac{n\bar{x}_n - n}{(1-\theta)^2},$$

so that

$$-\mathbb{E} \frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta) = \frac{n}{\theta^2} \qquad \frac{n/\theta - n}{(1-\theta)^2}$$

$$= n\left[\frac{(1-\theta)^2 + \theta - \theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= n\left[\frac{1 - 2\theta + \theta^2 + \theta - \theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= \frac{n}{\theta^2(1-\theta)}.$$

③ If $\tilde{\tau}(\underset{\sim}{X})$ is unbiased for θ, $\frac{\partial}{\partial\theta}\mathbb{E}_\theta\tilde{\tau}(\underset{\sim}{X}) = 1$, so the CRLB is

$$\text{CRLB} = \frac{\theta^2(1-\theta)}{n}$$

④ Recall: The UMVUE for θ is

$$\hat{\tau} = \binom{T-2}{n-2} \Big/ \binom{T-1}{n-1},$$

[set $k=1$ in $\tau(\theta) = \theta(1-\theta)^{k-1}$ from lecture 6]

where $T = \sum_{i=1}^n X_i$.

$$\text{Var}\,\hat{\tau} \geq \frac{\theta^2(1-\theta)}{n}.$$

⑤ If $\tilde{\tau}(\underset{\sim}{X})$ is unbiased for $1/\theta$, we have

$$\frac{\partial}{\partial\theta}\mathbb{E}_\theta\tilde{\tau}(\underset{\sim}{X}) = \frac{\partial}{\partial\theta}\left(\frac{1}{\theta}\right) = -\frac{1}{\theta^2},$$

so

$$\text{Var}\,\tilde{\tau}(\underset{\sim}{X}) \geq \frac{\left[-1/\theta^2\right]^2}{\left[n/\theta^2(1-\theta)\right]} = \frac{(1-\theta)}{n\theta^2}$$

⑥ Try $\bar{X}_n$: $\mathbb{E}\bar{X}_n = \frac{1}{\theta}$ and it's a function of a comp. suff.

Also, $\text{Var}\,\bar{X}_n = \frac{1}{n}\frac{1-\theta}{\theta^2}$,

so $\bar{X}_n$ achieves the CRLB for estimating $1/\theta$.

$$\boxed{\underset{\sim}{X} \overset{\text{ind}}{\sim} F(x;\mu) = \left[1 + e^{-(x-\mu)}\right]^{-1}, \quad f(x;\mu) = e^{-(x-\mu)} / \left[1 + e^{-(x-\mu)}\right]^2 .}$$

① Find $I(\mu)$.

$$f(\underset{\sim}{x};\mu) = e^{-\sum_{i=1}^n (x_i-\mu)} \prod_{i=1}^n \left[1 + e^{-(x_i-\mu)}\right]^{-2}$$

$$\log f(\underset{\sim}{x};\mu) = -\sum_{i=1}^n (x_i-\mu) - 2 \sum_{i=1}^n \log\left(1 + e^{-(x_i-\mu)}\right)$$

$$S(\mu;\underset{\sim}{X}) = \frac{\partial}{\partial\mu} \log f(\underset{\sim}{x};\mu) = n - 2 \sum_{i=1}^n \underbrace{\frac{e^{-(x_i-\mu)}}{1+e^{-(x_i-\mu)}}}_{1-F(x_i;\mu)}$$

$$= 2 \sum_{i=1}^n \left(F(x_i;\mu) - \frac{1}{2}\right)$$

Now we have

$$\operatorname{Var} S(\mu;\underset{\sim}{X}) = 4\, n \operatorname{Var}\Big(\frac{1}{2} - \underbrace{F(x_i;\mu)}_{\sim \text{Unif}(0,1)}\Big) = 4n\left(\frac{1}{12}\right) = \frac{n}{3}.$$

② For any unbiased estimator $\hat{\mu}$ of μ, we have

$$\operatorname{Var} \hat{\mu} \geq \frac{3}{n}.$$

③ Does $\bar{X}_n$ achieve the CRLB?

Well $\mathbb{E}\bar{X}_n = \mu$, but $\operatorname{Var} \bar{X}_n = \frac{1}{n}\frac{\pi^2}{3} > \frac{3}{n}$

Show that $\operatorname{Var} X_1 = \frac{\pi^2}{3}$.

$$\boxed{\frac{\pi^2}{3} = 3.29.}$$

$$\boxed{\begin{array}{l}\text{An unbiased estimator } \tilde{\tau}(\underset{\sim}{X}) \text{ of } \tau(\theta) \text{ attains the CRLB iff} \\ \quad S(\theta;\underset{\sim}{X}) = a(\theta)\left[\tilde{\tau}(\underset{\sim}{X}) - \tau(\theta)\right] \\ \text{for some constant } a(\theta) \text{ depending on } \theta.\end{array}}$$

Proof: $\tilde{\tau}(\underset{\sim}{X})$ attains the CRLB iff $\operatorname{corr}(\tilde{\tau}(\underset{\sim}{X}), S(\theta;\underset{\sim}{X})) = \pm 1$, since

$$\operatorname{Var} \tilde{\tau}(\underset{\sim}{X}) = \frac{\left[\frac{\partial}{\partial\theta} \mathbb{E}_\theta \tilde{\tau}(\underset{\sim}{X})\right]^2}{I(\theta)} = \frac{\left[\operatorname{Cov}(\tilde{\tau}(\underset{\sim}{X}), S(\theta;\underset{\sim}{X}))\right]^2}{\operatorname{Var} S(\theta;\underset{\sim}{X})}$$

$$\iff \operatorname{Corr}\left(\tilde{\tau}(\underset{\sim}{X}), S(\theta;\underset{\sim}{X})\right) = \pm 1.$$

Thm 4.5.7 in CB $\left(\operatorname{Corr}(X,Y) = \pm 1 \iff \exists\, a \neq 0, b \text{ s.t. } Y = aX + b \text{ w.p. } 1\right)$ gives that

$\operatorname{Corr}(\tilde{\tau}(\underset{\sim}{X}), S(\theta;\underset{\sim}{X})) = \pm 1$ iff $\exists$ $a(\theta)$ and $b(\theta)$ st

$$S(\theta;\underset{\sim}{X}) = a(\theta)\,\tilde{\tau}(\underset{\sim}{X}) + b(\theta) \quad (\text{w.p. } 1)$$

Now, $\mathbb{E}\, S(\theta;\underset{\sim}{X}) = 0$ and $\mathbb{E}\,\tilde{\tau}(\underset{\sim}{X}) = \tau(\theta)$, which gives

$$\mathbb{E}\left[a(\theta)\,\tilde{\tau}(\underset{\sim}{X}) + b(\theta)\right] = a(\theta)\,\tau(\theta) + b(\theta) = 0$$

$$\iff b(\theta) = -a(\theta)\,\tau(\theta).$$

So we have

$$S(\theta;\underset{\sim}{X}) = a(\theta)\left[\tilde{\tau}(\underset{\sim}{X}) - \tau(\theta)\right]$$

$\iff \tau(\underset{\sim}{X})$ achieves the CRLB.

$\boxed{\text{Attainment of CRLB for } X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(\lambda), \ \bar{X}_n \text{ for } \lambda}$

We have $f(x;\lambda) = \frac{1}{\lambda} e^{-x/\lambda} \mathbb{1}(x > 0)$, so

$$L(\lambda; \underset{\sim}{x}) = \left(\frac{1}{\lambda}\right)^n e^{-\sum_{i=1}^n x_i / \lambda}$$

$$\ell(\lambda; \underset{\sim}{x}) = -n \log \lambda - \frac{1}{\lambda} \sum_{i=1}^n x_i$$

$$S(\lambda; \underset{\sim}{x}) = \frac{\partial}{\partial \lambda} \ell(\lambda; \underset{\sim}{x}) = \frac{-n}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^n X_i = \frac{-n}{\lambda} + \frac{n}{\lambda^2} \bar{X}_n$$

Note that $\mathbb{E}\bar{X}_n = \lambda$. And we can write

$$S(\lambda; \underset{\sim}{x}) = \frac{n}{\lambda^2}\left[\bar{X}_n - \lambda\right],$$

so by the attainment theorem, $\bar{X}_n$ achieves the CRLB.

Note that $I(\lambda) = \text{Var}\, S(\lambda; \underset{\sim}{x}) = \frac{n^2}{\lambda^4} \text{Var}\, \bar{X}_n = \frac{n^2}{\lambda^4} \frac{\lambda^2}{n} = \frac{n}{\lambda^2}$

so that the CRLB for unbiased estimators of λ is $\frac{\lambda^2}{n}$.

We have $\text{Var}\, \bar{X}_n = \frac{\lambda^2}{n}$, so it achieves the CRLB.

$$\underset{\sim}{X} \overset{ind}{\sim} \text{Normal}(\mu, \sigma). \quad \text{Find Fisher Inf and CRLB for unbiased estimators of}$$

$$\tau(\mu,\sigma) = \sigma \qquad \tau(\mu,\sigma) = \mu + 3\sqrt{\sigma}$$

$$L(\mu,\sigma;\underset{\sim}{x}) = (2\pi)^{-n/2}\,\sigma^{-n/2}\exp\left[-\frac{1}{2\sigma}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$\ell(\mu,\sigma;\underset{\sim}{x}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma - \frac{1}{2\sigma}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial\mu}\ell(\mu,\sigma;\underset{\sim}{x}) = \frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial\sigma}\ell(\mu,\sigma;\underset{\sim}{x}) = -\frac{n}{2\sigma} + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

So we have

$$S(\mu,\sigma;\underset{\sim}{x}) = \begin{bmatrix} \frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\mu) \\ \\ -\frac{n}{2\sigma} + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix}.$$

Then

$$I(\mu,\sigma) = \text{Cov}\left(S(\mu,\sigma;\underset{\sim}{X})\right) = \begin{bmatrix} \frac{n}{\sigma} & 0 \\ \\ 0 & \frac{n}{2\sigma^2} \end{bmatrix},$$

Since

$$\frac{1}{4\sigma^4}\, n\, \text{Var}\left[(X_1-\mu)^2\right] = \frac{n}{4\sigma^4}\left[\overbrace{\mathbb{E}(X_1-\mu)^4}^{3\sigma^2} - \overbrace{\left(\mathbb{E}(X_1-\mu)^2\right)^2}^{\sigma^2}\right] = \frac{n}{4\sigma^4}\left[3\sigma^2 - \sigma^2\right] = \frac{n}{2\sigma^2}$$

(i) Now, for $\tau(\mu, \sigma) = \sigma$ we have

$$\begin{bmatrix} \frac{\partial}{\partial \mu} \tau(\mu, \sigma) \\ \\ \frac{\partial}{\partial \sigma} \tau(\mu, \sigma) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}$$

so that the CRLB for unbiased estimators of $\tau(\mu, \sigma) = \sigma$ is

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sigma}{n} & 0 \\ \\ 0 & \frac{2\sigma^2}{n} \end{bmatrix} \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = \frac{2\sigma^2}{n}.$$

Note that $\text{Var}\, S_n^2 = \frac{2\sigma^2}{n-1}$, since

$$\frac{(n-1)S_n^2}{\sigma} \sim \chi^2_{n-1} \quad \text{so} \quad \text{Var}\left[\frac{(n-1)S_n^2}{\sigma}\right] = 2(n-1)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^2} \text{Var}\, S_n^2 = 2(n-1)$$

$$\Rightarrow \text{Var}\, S_n^2 = \frac{2\sigma^2}{n-1}.$$

So S_n^2 does not achieve the CRLB when μ is unknown.

(ii) For $\tau(\mu, \sigma) = \mu + 3\sqrt{\sigma}$ we have

$$\frac{\partial}{\partial \mu} \tau(\mu, \sigma) = 1 \qquad \frac{\partial}{\partial \sigma} \tau(\mu, \sigma) = \frac{3}{2\sqrt{\sigma}}$$

so the CRLB for unbiased estimators of $\mu + 3\sqrt{\sigma}$ is

$$\begin{bmatrix} 1 & \frac{3}{2\sqrt{\sigma}} \end{bmatrix} \begin{bmatrix} \frac{\sigma}{n} & 0 \\ 0 & \frac{2\sigma^2}{n} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2\sqrt{\sigma}} \end{bmatrix} = \frac{\sigma}{n} + \frac{9}{4\sigma} \frac{2\sigma^2}{n}$$

$$= \frac{\sigma}{n}\left[1 + \frac{9}{2}\right]$$

$$= \frac{\sigma}{n}\left(\frac{11}{2}\right)$$