STAT 713 sp 2023 Lec 09 slides

Asymptotic distributions of estimators

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

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Recall the central limit theorem and the delta method:

Theorem (Central limit theorem, cf. Thm 5.5.15 in CB)

For $X_1, \dots, X_n$ iid, $\hat{\tau}_n = n^{-1} \sum_{i=1}^n h(X_i)$, $\tau = \mathbb{E} h(X_1)$, with $\vartheta = \operatorname{Var} h(X_1) < \infty$, we have

$$\sqrt{n}(\hat{\tau}_n - \tau) \xrightarrow{D} \text{Normal}(0, \vartheta)$$

as $n \to \infty$.

Theorem (Delta method, cf. Thm 5.5.24 in CB)

If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} \text{Normal}(0, \sigma^2)$ as $n \to \infty$, then

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{D} \text{Normal}(0, [g'(\theta)]^2 \sigma^2)$$

as $n \to \infty$, provided $g'(\theta)$ exists and is not 0.

The variance in a limiting Normal distribution is called the *asymptotic variance*.

Corollary (Asymptotic distribution of kth moment)

Let $X_1, \dots, X_n$ be iid with $m_{2k} = \mathbb{E}X_1^{2k} < \infty$ and let $\hat{m}_k = n^{-1}\sum_{i=1}^n X_i^k$ and $m_k = \mathbb{E}X_1^k$. Then

$$\sqrt{n}(\hat{m}_k - m_k) \xrightarrow{D} \text{Normal}(0, m_{2k} - m_k^2)$$

as $n \to \infty$.

Apply CLT in prev. slide with $h(x) = x^k$, noting that $\text{Var}\, X_1^k = \mathbb{E}X_1^{2k} - (\mathbb{E}X_1^k)^2$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$. Give the asympt. behavior of $\hat{m}_2$.

Asymptotic behavior of sample quantiles

Let $X_1, \dots, X_n$ be iid with cont. pdf f and pth quantile q_p. Set $\hat{q}_{np} = X_{(\lceil np \rceil)}$. Then

$$\sqrt{n}(\hat{q}_{np} - q_p) \xrightarrow{\text{D}} \text{Normal}\left(0, \frac{p(1-p)}{[f(q_p)]^2}\right)$$

as $n \to \infty$, provided $f(q_p) > 0$.

Exercise: Give heuristics of proof.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\theta) = \frac{1}{2\lambda} e^{-|x-\mu|/\lambda}$ with $\mu \in \mathbb{R}$ and $\lambda > 0$.

Give the asymptotic behavior of $\sqrt{n}(\hat{q}_{n,0.5} - q_{0.5})$ as $n \to \infty$.

Theorem (Asymptotic distribution of the MLE)

Let $X_1, \dots, X_n$ be iid with cdf $F(x;\theta)$ and let $\hat{\theta}_n$ be the MLE for θ. Suppose

1. *the support of $F(\cdot;\theta)$ does not depend on θ.*
2. *the score function exists and has finite mean.*
3. *the true value of θ lies in the interior of Θ.*
4. *conditions (A5) and (A6) from pg. 516 in CB hold.*

Then, for a continuous function τ, provided $\tau'(\theta)$ exists and is not zero, we have

$$\sqrt{n}(\tau(\hat{\theta}_n) - \tau(\theta)) \xrightarrow{D} \text{Normal}\left(0, \frac{[\tau'(\theta)]^2}{I_1(\theta)}\right),$$

as $n \to \infty$, where $I_1(\theta) = \text{Var}_\theta\left(\frac{\partial}{\partial\theta}\log f(X_1;\theta)\right)$ is the Fisher inf based on $n = 1$.

Can interpret $I_1(\theta)$ as the expected curvature of $\ell(\theta; X_1)$ at θ.

The MLE achieves, asymptotically, the CRLB for unbiased estimators of $\tau(\theta)$.

Discuss: Heuristics of proof.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\theta, 1)$.

Give the asymptotic variance of $\sqrt{n}(\log \hat{\theta}_n - \log \theta)$, where $\hat{\theta}_n$ is the MLE of θ.

Asymptotic relative efficiency, cf. Defn 10.1.16 of CB

If two estimators W_n and V_n satisfy

$$\begin{aligned}\sqrt{n}(W_n - \tau(\theta)) &\xrightarrow{\text{D}} \text{Normal}(0, \sigma_W^2)\\ \sqrt{n}(V_n - \tau(\theta)) &\xrightarrow{\text{D}} \text{Normal}(0, \sigma_V^2) \quad \text{as } n \to \infty,\end{aligned}$$

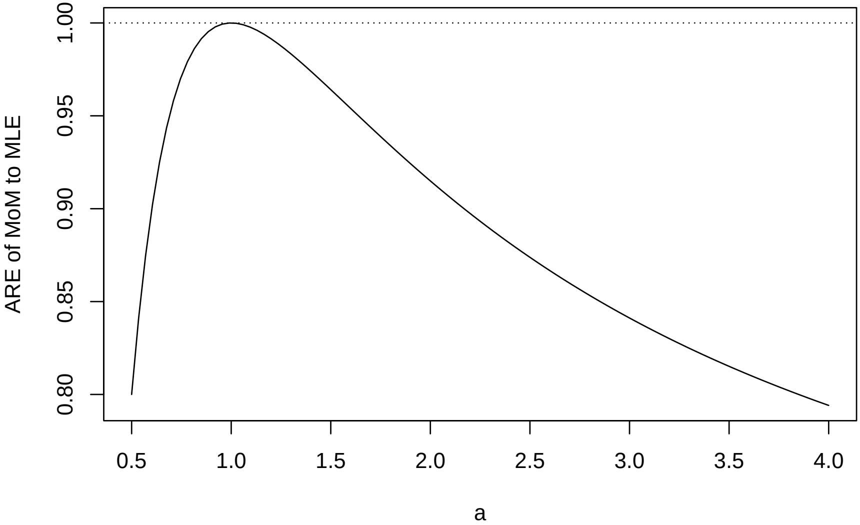
then the *asymptotic relative efficiency* of V_n with respect to W_n is defined as

$$\text{ARE}(V_n; W_n) = \sigma_W^2 / \sigma_V^2.$$

Exercise: Let $X_1, \dots, X_n$ be iid with density given by

$$f(x; b) = \left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^a\right] \mathbf{1}(x > 0),$$

for some $b > 0$, with $a > 0$ known. Find ARE of the MoM vs ML estimators of b.


ARE of MoM to MLE
1.00
0.95
0.90
0.85
0.80
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
a

Limiting efficiency, cf. Defn 10.1.11 of CB

Let $\hat{\tau}_n$ be a consistent estimator of $\tau = \tau(\theta)$ and suppose the CR condition holds. Then the *limiting efficiency* of $\hat{\tau}_n$ is defined as

$$\text{Leff}\,\hat{\tau}_n = \lim_{n\to\infty} \frac{[\frac{\partial}{\partial\theta}\tau(\theta)]^2/I_n(\theta)}{\text{Var}\,\hat{\tau}_n}$$

Moreover, τ_n is called *asymptotically efficient* if its limiting efficiency is equal to 1.

Limit of CRLB for unbiased estimators over the variance of the estimator.

Exercise: Find the limiting eff. of the MoM estimator of θ in $\text{Gamma}(\theta, 1)$.

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Theorem (Asymptotic joint distribution of moments)

Let $X_1, \dots, X_n$ *be iid and let* $\hat{m}_j = n^{-1} \sum_{i=1}^n X_i^j$ *and* $m_j = \mathbb{E} X_1^j$ *for* $j = 1, 2, \dots$ *Set* $\hat{\mathbf{m}} = (\hat{m}_1, \dots, \hat{m}_k)^T$ *and* $\mathbf{m} = (m_1, \dots, m_k)^T$ *and assume* $m_{2k} < \infty$. *Then*

$$n^{1/2}(\hat{\mathbf{m}} - \mathbf{m}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \mathbf{M})$$

as $n \to \infty$, *where* $\mathbf{M}_{ij} = m_{i+j} - m_i m_j$ *for* $1 \leq i, j \leq k$.

Theorem (Multivariate delta method)

Let $\sqrt{n}(\mathbf{Y}_n - \boldsymbol{\theta}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma})$ *as* $n \to \infty$. *Then*

$$\sqrt{n}[g(\mathbf{Y}_n) - g(\boldsymbol{\theta})] \xrightarrow{D} \text{Normal}\left(0, [\nabla g(\boldsymbol{\theta})]^T \boldsymbol{\Sigma}\, [\nabla g(\boldsymbol{\theta})]\right) \quad \text{as } n \to \infty,$$

provided $\nabla g(\boldsymbol{\theta})$ *exists and is not equal to zero.*

Exercise: For $X_1, \dots, X_n$ with $m_4 < \infty$, give asymptotic behavior of $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2)$, where $\sigma^2 = m_2 - m_1^2$ and $\hat{\sigma}_n^2 = \hat{m}_2 - \hat{m}_1^2$.

Theorem (Asymptotic distribution of the MLE, multidimensional θ)

Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} F(x;\theta)$ *and let* $\hat{\theta}_n$ *be the MLE for* $\theta \in \Theta \subset \mathbb{R}^d$. *Suppose*

1. *the support of* $F(\cdot;\theta)$ *does not depend on* θ.
2. *the score function exists and has finite mean.*
3. *the true value of* θ *lies in the interior of* Θ.
4. *conditions (A5) and (A6) from pg. 516 in CB hold.*

Then, for a continuous function τ, *provided* $\nabla\tau(\theta)$ *exists and is not zero, we have*

$$\sqrt{n}(\tau(\hat{\theta}_n) - \tau(\theta)) \xrightarrow{D} \text{Normal}\left(0, [\nabla\tau(\theta)]^T I_1^{-1}(\theta)[\nabla\tau(\theta)]\right),$$

as $n \to \infty$, *where* $I_1(\theta) = \mathbb{E}_\theta[S(\theta;X_1)S(\theta;X_1)^T]$ *is the Fisher inf based on* $n = 1$.

From now on we will call conditions 1–4 above the *MLE regularity conditions*.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} F_X(x; a, b) = 1 - \exp(-(x/b)^a)$, $x > 0$, $a, b > 0$.

Show the following:

1. $I_1(a, b) = \begin{bmatrix} 1.824/a^2 & -0.423/b \\ -0.423/b & a^2/b^2 \end{bmatrix}$.
2. $\text{ARE}(\hat{q}_{n,0.5}; \hat{\tau}_n) = 0.663$ where $\hat{\tau}_n$ is the MLE of the median.