Central limit theorem.
Delta.

STAT 713 sp 2023 Lec 09 slides

Asymptotic distributions of estimators

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

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Recall the central limit theorem and the delta method:

Theorem (Central limit theorem, cf. Thm 5.5.15 in CB)

For $X_1, \dots, X_n$ iid, $\hat{\tau}_n = n^{-1}\sum_{i=1}^n h(X_i)$, $\tau = \mathbb{E}h(X_1)$, with $\vartheta = \mathrm{Var}\, h(X_1) < \infty$, we have

$$\sqrt{n}(\hat{\tau}_n - \tau) \xrightarrow{D} \mathrm{Normal}(0, \vartheta)$$

as $n \to \infty$.

Theorem (Delta method, cf. Thm 5.5.24 in CB)

If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} \mathrm{Normal}(0, \sigma^2)$ as $n \to \infty$, then

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{D} \mathrm{Normal}(0, [g'(\theta)]^2\sigma^2)$$

as $n \to \infty$, provided $g'(\theta)$ exists and is not 0.

The variance in a limiting Normal distribution is called the *asymptotic variance*.

$h(x) = x^k$

Corollary (Asymptotic distribution of kth moment)

Let $X_1, \dots, X_n$ be iid with $m_{2k} = \mathbb{E}X_1^{2k} < \infty$ and let $\hat{m}_k = n^{-1}\sum_{i=1}^n X_i^k$ and $m_k = \mathbb{E}X_1^k$. Then

$$\sqrt{n}(\hat{m}_k - m_k) \xrightarrow{D} \text{Normal}(0, m_{2k} - m_k^2)$$

as $n \to \infty$.

$$\text{Var}\, X_1^k = \mathbb{E}(X_1^k)^2 - (\mathbb{E}\, X_1^k)^2 = \mathbb{E}\, X_1^{2k} - (\mathbb{E}\, X_1^k)^2 = m_{2k} - (m_k)^2$$

Apply CLT in prev. slide with $h(x) = x^k$, noting that $\text{Var}\, X_1^k = \mathbb{E}X_1^{2k} - (\mathbb{E}X_1^k)^2$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$. Give the asympt. behavior of $\hat{m}_2$.

$$\sqrt{n}(\hat{m}_2 - m_2) \xrightarrow{D} N(0, \underline{m_4 - m_2^2})$$

$$\sqrt{n}(\hat{m}_2 - \underline{\beta^2\alpha(\alpha+1)}) \xrightarrow{D} N(0, \underline{\qquad\qquad})$$

$$m_2 = \mathbb{E} X_1^2 = \mathrm{Var}\, X_1 + (\mathbb{E} X_1)^2 = \alpha\beta^2 + (\alpha\beta)^2$$

$$= \beta^2 \alpha(\alpha+1)$$

$$m_4 = \mathbb{E} X_1^4 = \int_0^\infty x^4 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{\Gamma(\alpha+4)\beta^{\alpha+4}}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty \frac{x^{(\alpha+4)-1}}{\Gamma(\alpha+4)\beta^{\alpha+4}} e^{-x/\beta}\, dx}_{=1}$$

$$= \frac{\beta^4(\alpha+3)(\alpha+2)(\alpha+1)\alpha\, \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$= \beta^4 (\alpha+3)(\alpha+2)(\alpha+1)\alpha$$

$$m_4 - m_2^2 = \beta^4(\alpha+3)(\alpha+2)(\alpha+1)\alpha - \left[\beta^2 \alpha(\alpha+1)\right]^2$$

$$= \beta^4 \alpha(\alpha+1)\left[(\alpha+3)(\alpha+2) - \alpha(\alpha+1)\right]$$

$$\sqrt{n}\left(\hat{m}_2 - \beta^2\alpha(\alpha+1)\right) \xrightarrow{D} N(0, \vartheta)$$

$$\vartheta = \beta^4 \alpha(\alpha+1)\left[(\alpha+3)(\alpha+2) - \alpha(\alpha+1)\right]$$

q_p = p^{th} quantile

Asymptotic behavior of sample quantiles

Let $X_1, \dots, X_n$ be iid with cont. pdf f and pth quantile q_p. Set $\hat{q}_{np} = X_{(\lceil np \rceil)}$. Then

$$\sqrt{n}(\hat{q}_{np} - q_p) \xrightarrow{\text{D}} \text{Normal}\left(0, \frac{p(1-p)}{[f(q_p)]^2}\right)$$

as $n \to \infty$, provided $f(q_p) > 0$.

Exercise: Give heuristics of proof.

$$P\left(\sqrt{n}(\hat{q}_{np} - q_p) \le x\right) = P\left(\hat{q}_{np} \le q_p + \frac{x}{\sqrt{n}}\right)$$

$$= P\left(X_{(\lceil np \rceil)} \le \xi_p + \frac{x}{\sqrt{n}} \right)$$

$$= P\left(N_{\xi_p + \frac{x}{\sqrt{n}}} \ge \lceil np \rceil \right)$$

$$= P\left(N_{\xi_p + \frac{x}{\sqrt{n}}} < \lceil np \rceil \right)$$

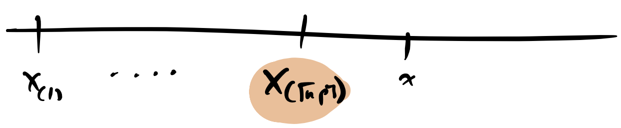
$$\Phi(x) = 1 - \Phi(-x)$$

$$\approx 1 - \Phi\left(\frac{\lceil np \rceil - n F_X(\xi_p + \frac{x}{\sqrt{n}})}{\sqrt{n F_X(\xi_p + \frac{x}{\sqrt{n}}) [1 - F_X(\xi_p + \frac{x}{\sqrt{n}})]}} \right)$$

$$= \Phi\left(\frac{n F_X(\xi_p + \frac{x}{\sqrt{n}}) - \lceil np \rceil}{\sqrt{n F_X(\xi_p + \frac{x}{\sqrt{n}}) [1 - F_X(\xi_p + \frac{x}{\sqrt{n}})]}} \right)$$

$$\approx \Phi\left(\frac{n \left(F_X(\xi_p) + \frac{x}{\sqrt{n}} f(\xi_p) \right) - \lceil np \rceil}{\sqrt{n \underbrace{F_X(\xi_p)}_{p} [1 - \underbrace{F_X(\xi_p)}_{p}]}} \right)$$

$$= \Phi\left(\frac{np + \sqrt{n}\, x f(\xi_p) - \lceil np \rceil}{\sqrt{n\, p(1-p)}} \right)$$



$$\approx \Phi\left(\frac{x}{\sqrt{\frac{p(1-p)}{f^2(\xi_p)}}} \right)$$

$$X_{(\lceil np \rceil)} \le x \iff \underbrace{\#\{\text{obs} \le x\}}_{N_x} \ge \lceil np \rceil$$

$$N_x \sim \text{Binomial}(n, F_X(x))$$

$$\approx \text{Normal}\left(n F_X(x),\ n F_X(x) [1 - F_X(x)] \right)$$

$$P\left(\sqrt{n}\left(\hat{\xi}_{np} - \xi_p\right) \le x\right) \approx \Phi\left(x \Big/ \sqrt{\frac{p(1-p)}{f^2(\xi_p)}}\right)$$

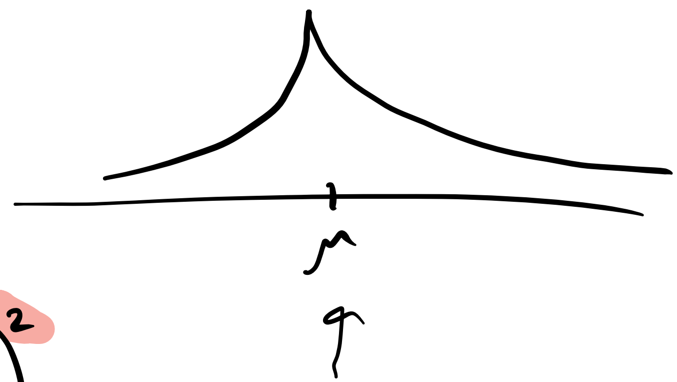
$\uparrow$ cdf of $N\left(0, \frac{p(1-p)}{f^2(\xi_p)}\right)$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\theta) = \boxed{\frac{1}{2\lambda} e^{-|x-\mu|/\lambda}}$ with $\mu \in \mathbb{R}$ and $\lambda > 0$.

Give the asymptotic behavior of $\sqrt{n}(\hat{q}_{n,0.5} - q_{0.5})$ as $n \to \infty$.

$$\sqrt{n}\left(\hat{\xi}_{n,0.5} - \xi_{0.5}\right) \xrightarrow{D} N(0, \vartheta)$$

$\uparrow$ $X_{(\lceil \frac{1}{2} n \rceil)}$



$$\vartheta = \frac{\frac{1}{2}\left(1 - \frac{1}{2}\right)}{\left(\frac{1}{2\lambda} e^{-|\underset{\xi_{0.5}}{\mu} - \mu|/\lambda}\right)^2}$$

$$= \lambda^2$$

$$\Rightarrow \sqrt{n}\left(X_{(\lceil \frac{1}{2} n \rceil)} - \mu\right) \xrightarrow{D} N(0, \lambda^2)$$

Theorem (Asymptotic distribution of the MLE)

$\theta \in \Theta \subset \mathbb{R}$

Let $X_1, \dots, X_n$ *be iid with cdf* $F(x;\theta)$ *and let* $\hat{\theta}_n$ *be the MLE for* θ*. Suppose*

1. *the support of* $F(\cdot;\theta)$ *does not depend on* θ*.*
2. *the score function exists and has finite mean.*
3. *the true value of* θ *lies in the interior of* Θ*.*
4. *conditions (A5) and (A6) from pg. 516 in CB hold.*

Then, for a continuous function τ*, provided* $\tau'(\theta)$ *exists and is not zero, we have*

$$\sqrt{n}(\tau(\hat{\theta}_n) - \tau(\theta)) \xrightarrow{D} \text{Normal}\left(0, \frac{[\tau'(\theta)]^2}{I_1(\theta)}\right),$$

as $n \to \infty$*, where* $I_1(\theta) = \text{Var}_\theta\left(\frac{\partial}{\partial\theta}\log f(X_1;\theta)\right)$ *is the Fisher inf based on* $n = 1$*.*

Can interpret $I_1(\theta)$ as the expected curvature of $\ell(\theta; X_1)$ at θ.

The MLE achieves, asymptotically, the CRLB for unbiased estimators of $\tau(\theta)$.

Discuss: Heuristics of proof.

$X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta)$, let $f(\underset{\sim}{x};\theta)$ be joint pdf/pmf.

Let θ_0 be the true parameter value.

First show: $\sqrt{n}(\hat{\theta}_n - \theta) \to N\left(0, \frac{1}{I_1(\theta)}\right)$

$$s(\theta;\underset{\sim}{x}) = \frac{\partial}{\partial\theta} \log f(\underset{\sim}{x};\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(x_i;\theta)$$

Hessian

$$\frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta) = H(\theta;\underset{\sim}{x})$$

Do Taylor Expansion

$$0 = s(\hat{\theta}_n;\underset{\sim}{x}) \approx s(\theta_0;\underset{\sim}{x}) + \left[\frac{\partial}{\partial\theta} s(\theta;\underset{\sim}{x})\right]\Bigg|_{\theta_0} (\hat{\theta}_n - \theta_0)$$

$$\Leftrightarrow \quad (\hat{\theta}_n - \theta) \approx \frac{s(\theta_0;\underset{\sim}{x})}{-\left[\frac{\partial}{\partial\theta} s(\theta;\underset{\sim}{x})\right]\Big|_{\theta_0}}$$

$$\Rightarrow \quad \sqrt{n}(\hat{\theta}_n - \theta) \approx \frac{\sqrt{n}\, s(\theta_0;\underset{\sim}{x})}{\left[-\frac{\partial^2}{\partial\theta^2} \log f(\underset{\sim}{x};\theta)\Big|_{\theta=\theta_0}\right]}$$

mean 0, variance $I_1(\theta_0)$

$$= \frac{\sqrt{n}\, \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial}{\partial\theta}\log f(x_i;\theta)\Big|_{\theta=\theta_0}\right] \xrightarrow{D} N(0, I_1(\theta_0))}{\underbrace{-\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial^2}{\partial\theta^2}\log f(x_i;\theta)\right]\Big|_{\theta=\theta_0}}_{\xrightarrow{a.s.} E_{\theta_0}\left[-\frac{\partial^2}{\partial\theta^2}\log f(x_i;\theta)\big|_{\theta=\theta_0}\right] = I_1(\theta_0)}}$$

$E_\theta\, s(\theta;x) = 0$

$$Var_{\theta_0}\left[\frac{\partial}{\partial\theta}\log f(x_i;\theta)\Big|_{\theta=\theta_0}\right] = I_1(\theta_0)$$

$$\longrightarrow N\left(0, \left[\frac{1}{I_1(\theta_0)}\right]^2 I_1(\theta_0)\right)$$

"

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \xrightarrow{D} N\left(0, \frac{1}{I_1(\theta_0)}\right)$$

Thm

$$\sqrt{n}\left(\tau(\hat{\theta}_n) - \tau(\theta_0)\right) \xrightarrow{D} N\left(0, \frac{[\tau'(\theta_0)]^2}{I_1(\theta_0)}\right)$$

$$f(x;\theta) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} \mathbb{1}(x>0)$$

Exercise: Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\theta, 1)$.

Give the asymptotic variance of $\sqrt{n}(\log \hat{\theta}_n - \log \theta)$, where $\hat{\theta}_n$ is the MLE of θ.

① $\sqrt{n}(\hat{\theta}_n - \theta) \longrightarrow \text{Normal}\left(0, \frac{1}{I_1(\theta)}\right)$

$$S(\theta; X_1) = \frac{\partial}{\partial\theta} \log f(X_1;\theta)$$

$$= \frac{\partial}{\partial\theta} \log\left(\frac{1}{\Gamma(\theta)} X_1^{\theta-1} e^{-X_1}\right)$$

$$= \frac{\partial}{\partial\theta}\left[-\log\Gamma(\theta) - (\theta-1)\log X_1 - X_1\right]$$

$$= -\frac{\Gamma'(\theta)}{\Gamma(\theta)} - \log(x_1)$$

$$= -\psi(\theta) - \log(x_1)$$

"digamma"

$$I_1(\theta) = \mathbb{E}_\theta\left[-\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta)\right]$$

$$= -\mathbb{E}_\theta\left[\frac{\partial}{\partial\theta}\left(-\psi(\theta) - \log(x_1)\right)\right]$$

$$= \psi'(\theta)$$

"trigamma"

$$\Rightarrow \quad \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} N\left(0, \frac{1}{\psi'(\theta)}\right)$$

② $$\sqrt{n}(\log\hat{\theta}_n - \log\theta) \longrightarrow N\left(0, \frac{1}{\theta^2}\frac{1}{\psi'(\theta)}\right)$$

$$\tau(\theta) = \log\theta$$

$$\tau'(\theta) = \frac{1}{\theta}$$

Asymptotic relative efficiency, cf. Defn 10.1.16 of CB

If two estimators W_n and V_n satisfy

$$\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{\text{D}} \text{Normal}(0, \sigma^2_W)$$
$$\sqrt{n}(V_n - \tau(\theta)) \xrightarrow{\text{D}} \text{Normal}(0, \sigma^2_V),$$

as $n \to \infty$, then the *asymptotic relative efficiency* of W_n versus V_n is defined as

$$\text{ARE}(V_n; W_n) = \sigma^2_W / \sigma^2_V.$$

Exercise: Let $X_1, \dots, X_n$ be iid with density given by

$$f(x; b) = \left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^{a}\right] \mathbf{1}(x > 0),$$

for some $\boxed{b > 0,}$ with $a > 0$ known. Find ARE of the MoM vs ML estimators of b.

$$\boxed{\begin{aligned} m_1 &= b\,\Gamma\left(1+\tfrac{1}{a}\right) \\ m_2 &= b^2\,\Gamma\left(1+\tfrac{2}{a}\right)\end{aligned}}$$

MoM: $\bar{b}_n = \dfrac{\hat{m}_1}{\Gamma(1+\frac{1}{a})}$

$b = m_1 / \Gamma(1+\frac{1}{a})$

$$\sqrt{n}(\hat{m}_1 - m_1) \xrightarrow{D} N\Big(0, \underbrace{m_2 - m_1^2}_{\sigma^2 =\; b^2\Gamma(1+\frac{2}{a}) - b^2\Gamma^2(1+\frac{1}{a})}\Big)$$

$$\sqrt{n}\left(g(\hat{m}_1) - g(m_1)\right) \xrightarrow{D} N\left(0, [g'(m_1)]^2\sigma^2\right)$$

$$b = g(m_1) = \frac{m_1}{\Gamma(1+\frac{1}{a})}, \quad g'(m_1) = \frac{1}{\Gamma(1+\frac{1}{a})}$$

$$\sqrt{n}(\bar{b}_n - b) \xrightarrow{D} N\left(0, \frac{1}{\Gamma^2(1+\frac{1}{a})}\left(b^2\Gamma(1+\tfrac{2}{a}) - b^2\Gamma^2(1+\tfrac{1}{a})\right)\right)$$

$$\text{"} \quad N\left(0, b^2\left(\frac{\Gamma(1+\frac{2}{a})}{\Gamma^2(1+\frac{1}{a})} - 1\right)\right)$$

MLE

$$\sqrt{n}(\hat{b}_n - b) \xrightarrow{D} N\left(0, \frac{1}{I_1(b)}\right)$$

$$f(x;b) = \left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\exp\left[-\left(\frac{x}{b}\right)^{a}\right]\mathbf{1}(x>0),$$

$$\log f(x;b) = \log a - \log b + (a-1)\log x - (a-1)\log b - \left(\frac{x}{b}\right)^a$$

$$\frac{\partial}{\partial b}\log f(x;b) = -\frac{1}{b} - \frac{(a-1)}{b} - a\left(\frac{x}{b}\right)^{a-1}\left(-\frac{x}{b^2}\right)$$

$$= -\frac{a}{b} + \frac{a}{b}\left(\frac{x}{b}\right)^a$$

$$s(b;X_1) = -\frac{a}{b} + \frac{a}{b}\left(\frac{X_1}{b}\right)^a$$

$$I_1(b) = \mathrm{Var}\left(s(b;X_1)\right) = \left(\frac{a}{b}\right)^2 \mathrm{Var}\left(\underbrace{\left(\frac{X_1}{b}\right)^a}_{\sim \text{Exponential }(1)}\right) = \frac{a^2}{b^2}.$$

So

$$\sqrt{n}\left(\hat{b}_n - b\right) \xrightarrow{D} \text{Normal}\left(0, \frac{b^2}{a^2}\right)$$

$$\mathrm{ARE}(V_n; W_n) = \sigma_W^2/\sigma_V^2.$$

$$\mathrm{ARE}\left(\bar{b}_n; \hat{b}_n\right) = \frac{b^2/a^2}{b^2\left(\frac{\Gamma\left(1+\frac{2}{a}\right)}{\Gamma^2\left(1+\frac{1}{a}\right)} - 1\right)}$$

$$= \frac{1}{a^2\left(\frac{\Gamma\left(1+\frac{2}{a}\right)}{\Gamma^2\left(1+\frac{1}{a}\right)} - 1\right)}$$

Limiting efficiency, cf. Defn 10.1.11 of CB

Let $\hat{\tau}_n$ be a consistent estimator of $\tau = \tau(\theta)$ and suppose the CR condition holds. Then the *limiting efficiency* of $\hat{\tau}_n$ is defined as

$$\text{Leff}\,\hat{\tau}_n = \lim_{n\to\infty} \frac{[\frac{\partial}{\partial\theta}\tau(\theta)]^2 / I_n(\theta)}{\text{Var}\,\hat{\tau}_n}$$

$\leftarrow$ CRLB

Moreover, τ_n is called *asymptotically efficient* if its limiting efficiency is equal to 1.

Limit of CRLB for unbiased estimators over the variance of the estimator.

$\tau(\theta)=\theta$ $\qquad \alpha\beta^2$

Exercise: Find the limiting eff. of the MoM estimator of θ in Gamma$(\theta, 1)$.

M.M: $\bar{\theta}_n = \hat{m}_1$ $\qquad \text{Var}\,\bar{\theta}_n = \text{Var}\,\hat{m}_1 = \text{Var}\,\bar{X}_n = \dfrac{\theta}{n}$ $\qquad m_1 = \theta$

θ

CRLB: $L(\theta; \underline{X}) = \prod_{i=1}^{n} \frac{1}{\Gamma(\theta)} x_i^{\theta-1} e^{-x_i} = (\Gamma(\theta))^{-n} (\prod x_i)^{\theta-1} e^{-\sum x_i}$

$$\ell(\theta;\underline{x}) = -n\log\Gamma(\theta) + (\theta-1)\sum\log x_i - \sum x_i$$

$$s(\theta;\underline{x}) = -n\underbrace{\frac{\Gamma'(\theta)}{\Gamma(\theta)}}_{=\psi(\theta)} + \sum\log x_i$$

$$I_n(\theta) = -\mathbb{E}_\theta\left[\frac{\partial}{\partial\theta} s(\theta;\underline{x})\right]$$

$$= -\mathbb{E}_\theta\left[-n\,\psi'(\theta)\right]$$

$$= n\,\psi'(\theta)$$

$\tau(\theta) = \theta$ so $\frac{\partial}{\partial\theta}\tau(\theta) = 1$

$$\text{CRLB} = \frac{1}{n\psi'(\theta)}$$

$$\text{Leff}\ \bar{\theta}_n = \lim_{n\to\infty}\frac{1/[n\psi'(\theta)]}{\theta/n} = \frac{1}{\theta\psi'(\theta)}$$

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$$M_{ij} = m_{i+j} - m_i m_j$$

$$\sqrt{n}\left(\begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \\ \vdots \\ \hat{m}_d \end{bmatrix} - \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_d \end{bmatrix}\right) \xrightarrow{D} N\left(\underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{d\times 1}, \underbrace{\begin{bmatrix} m_2 - m_1^2 & \overset{(2,1)}{m_3 - m_2 m_1} & \cdots \\ m_3 - m_2 m_1 & m_4 - m_2^2 & \\ \vdots & & \ddots \end{bmatrix}}_{d\times d}\right)$$

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$$

$$\sqrt{n}(\hat{m}_1 - m_1) \xrightarrow{D} N(0, m_2 - m_1^2)$$

Theorem (Asymptotic joint distribution of moments)

Let $X_1, \ldots, X_n$ *be iid and let* $\hat{m}_j = n^{-1}\sum_{i=1}^n X_i^j$ *and* $m_j = \mathbb{E}X_1^j$ *for* $j = 1, 2, \ldots$ *Set* $\hat{\mathbf{m}} = (\hat{m}_1, \ldots, \hat{m}_k)^T$ *and* $\mathbf{m} = (m_1, \ldots, m_k)^T$ *and assume* $m_{2k} < \infty$. *Then*

$$n^{1/2}(\hat{\mathbf{m}} - \mathbf{m}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \mathbf{M})$$

as $n \to \infty$, *where* $\mathbf{M}_{ij} = m_{i+j} - m_i m_j$ *for* $1 \leq i, j \leq k$.

Theorem (Multivariate delta method)

Let $\sqrt{n}(\underset{d\times 1}{\mathbf{Y}_n} - \underset{d\times 1}{\boldsymbol{\theta}}) \xrightarrow{D} \text{Normal}(\mathbf{0}, \underset{d\times d}{\boldsymbol{\Sigma}})$ *as* $n \to \infty$. *Then*
sandwich

$$\sqrt{n}[g(\mathbf{Y}_n) - g(\boldsymbol{\theta})] \xrightarrow{D} \text{Normal}\left(0, [\nabla g(\boldsymbol{\theta})]^T \boldsymbol{\Sigma}\, [\nabla g(\boldsymbol{\theta})]\right) \quad \text{as } n \to \infty,$$

provided $\nabla g(\boldsymbol{\theta})$ *exists and is not equal to zero.*

Exercise: For $X_1, \ldots, X_n$ iid with $m_4 < \infty$, give asymptotic behavior of $\boxed{\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2),}$ where $\sigma^2 = m_2 - m_1^2$ and $\hat{\sigma}_n^2 = \hat{m}_2 - \hat{m}_1^2$.

$$\sqrt{n}\left(\hat{\sigma}_n^2 - \sigma^2\right) = \sqrt{n}\left(g(\hat{m}_1, \hat{m}_2) - g(m_1, m_2)\right),$$

$$g(m_1, m_2) = m_2 - m_1^2$$

$$\sqrt{n}\left(\begin{pmatrix}\hat{m}_1\\ \hat{m}_2\end{pmatrix} - \begin{pmatrix}m_1\\ m_2\end{pmatrix}\right) \xrightarrow{D} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} m_2 - m_1^2 & m_3 - m_2 m_1 \\ m_3 - m_2 m_1 & m_4 - m_2^2\end{pmatrix}\right)$$

$$\sqrt{n}\left(\hat{\sigma}_n^2 - \sigma^2\right) \xrightarrow{D} N(0, \vartheta),$$

where,

$$\vartheta = \left[\nabla g(m_1, m_2)\right]^T M \, \nabla g(m_1, m_2).$$

$$\nabla g(m_1, m_2) = \begin{bmatrix} -2m_1 \\ 1 \end{bmatrix}.$$

So

$$\vartheta = \begin{pmatrix} -2m_1 \\ 1 \end{pmatrix}^T \begin{pmatrix} m_2 - m_1^2 & m_3 - m_2 m_1 \\ m_3 - m_2 m_1 & m_4 - m_2^2 \end{pmatrix} \begin{pmatrix} -2m_1 \\ 1 \end{pmatrix}$$

$$= 4m_1^2 (m_2 - m_1^2) - 2(2m_1)(m_3 - m_2 m_1) + m_4 - m_2^2.$$

One dimension $\longrightarrow$ $\sqrt{n}(\tau(\hat{\theta}_n) - \tau(\theta)) \xrightarrow{D} \text{Normal}\left(0, \frac{[\tau'(\theta)]^2}{I_1(\theta)}\right)$,

Theorem (Asymptotic distribution of the MLE, multidimensional θ)

Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} F(x;\theta)$ *and let* $\hat{\theta}_n$ *be the MLE for* $\theta \in \Theta \subset \mathbb{R}^d$. *Suppose*

1. *the support of* $F(\cdot;\theta)$ *does not depend on* θ.
2. *the score function exists and has finite mean.*
3. *the true value of* θ *lies in the interior of* Θ.
4. *conditions (A5) and (A6) from pg. 516 in CB hold.*

Then, for a continuous function τ, *provided* $\nabla\tau(\theta)$ *exists and is not zero, we have*

$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} \text{Normal}(0, I_1^{-1}(\theta))$

$$\sqrt{n}(\tau(\hat{\theta}_n) - \tau(\theta)) \xrightarrow{D} \text{Normal}\left(0, [\nabla\tau(\theta)]^T I_1^{-1}(\theta)[\nabla\tau(\theta)]\right),$$

as $n \to \infty$, *where* $I_1(\theta) = \mathbb{E}_\theta[S(\theta;X_1)S(\theta;X_1)^T]$ *is the Fisher inf based on* $n = 1$.

From now on we will call conditions 1–4 above the *MLE regularity conditions*.

solve for $q_{0.5}$:
$\frac{1}{2} = F_X(q_{0.5})$

Weibull.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} F_X(x; a, b) = 1 - \exp(-(x/b)^a)$, $x > 0$, $a, b > 0$.

Show the following:

1. $I_1(a, b) = \begin{bmatrix} 1.824/a^2 & -0.423/b \\ -0.423/b & a^2/b^2 \end{bmatrix}$.
2. $\text{ARE}(\hat{q}_{n,0.5}; \hat{\tau}_n) = 0.663$ where $\hat{\tau}_n$ is the MLE of the median.

$\hat{q}_{n,0.5} = X_{(\lceil 0.5 \cdot n \rceil)}$

Median $= \tau(a, b)$.

$\hat{\tau}_n = \tau(\hat{a}_n, \hat{b}_n)$, $\hat{a}_n, \hat{b}_n$ are MLEs.

$\sqrt{n}(\hat{q}_{n,0.5} - q_{0.5}) \to N(0, \vartheta_1)$

$$\sqrt{n}\left(\hat{\tau}_n - \tau\right) \xrightarrow{D} N\left(0, v_2\right)$$

Use delta method.