STAT 713 sp 2023 Lec 11 slides

Unbiased tests, uniformly most powerful tests, Neyman-Pearson and Karlin-Rubin

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

As we asked what makes a good estimator, we now ask: What makes a good test?

Every "sensible" test should have the following property.

Unbiasedness of a test of hypotheses

A test of $H_0\colon \theta \in \Theta_0$ vs $H_1\colon \theta \in \Theta_1$ with power function γ is called *unbiased* if

$$\gamma(\theta_1) \geq \gamma(\theta_0) \text{ for all } \theta_0 \in \Theta_0 \text{ and } \theta_1 \in \Theta_1.$$

An unbiased test is more likely to reject H_0 when $\theta \in \Theta_1$ than when $\theta \in \Theta_0$.

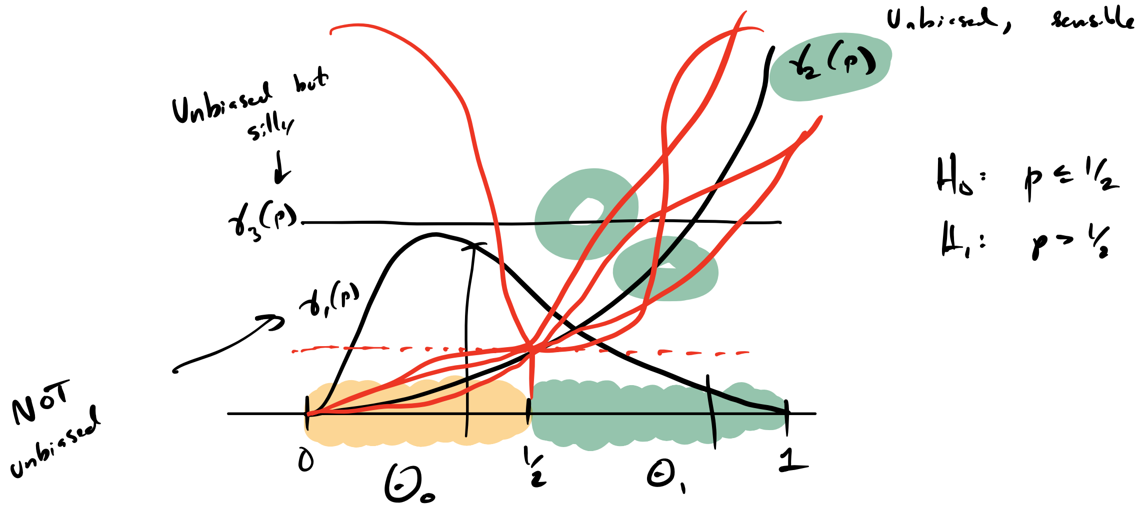
Exercise: For $Y \sim \text{Binom}(3, p)$, $H_0\colon p \leq 1/2$ v $H_1\colon p > 1/2$, check unbiasedness:

1. Reject H_0 if $Y = 1$.
2. Reject H_0 if $Y = 3$.
3. Flip a coin and reject H_0 if it is "heads".

$$\gamma_1(p) = P_p(Y=1) = \binom{3}{1} p^1 (1-p)^{3-1} = 3p(1-p)^2$$

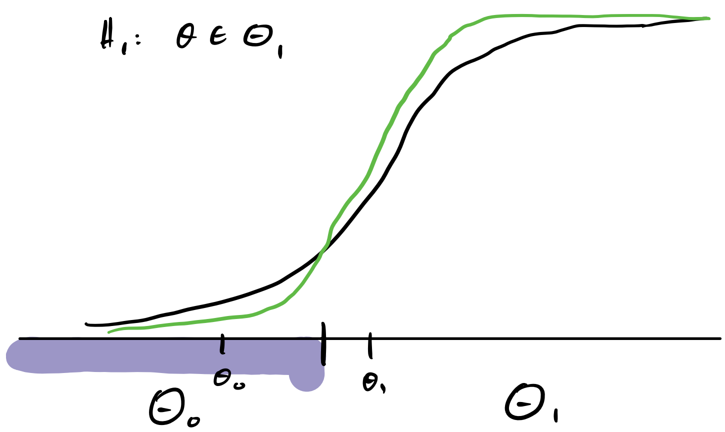
$$\gamma_2(p) = P_p(Y=3) = \binom{3}{3} p^3 (1-p)^{3-3} = p^3$$

$$\gamma_3(p) = \frac{1}{2}$$



$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1$



Consider comparing tests belonging to a class $\mathcal{C}$, for example

$$\mathcal{C}_\alpha = \{ \text{ all tests with level } \alpha \text{ } \}$$
$$\mathcal{C}_{\alpha,U} = \{ \text{ all tests with level } \alpha \text{ that are unbiased } \}.$$

Uniformly most powerful tests

Given a class $\mathcal{C}$ of tests of $H_0\colon \theta \in \Theta_0$ vs $H_1\colon \theta \in \Theta_1$, a test in $\mathcal{C}$ with power curve γ is the *uniformly most powerful (UMP)* test in $\mathcal{C}$ if

$$\gamma(\theta) \geq \tilde{\gamma}(\theta) \quad \text{for all } \theta \in \Theta_1,$$

where $\tilde{\gamma}$ is the power curve of any other test in $\mathcal{C}$.

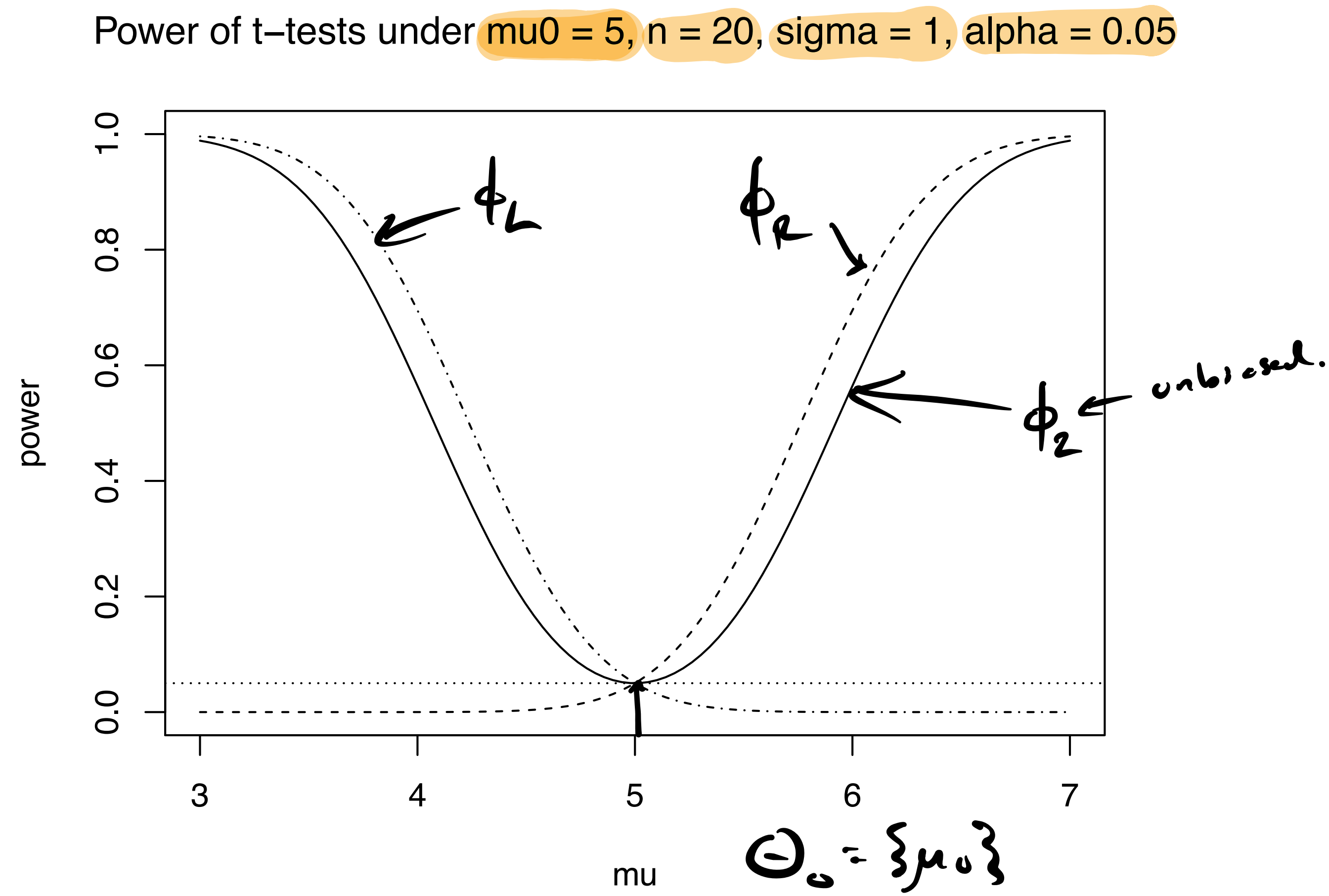
From now on we will refer by

1. *UMP test* to a test which is UMP in $\mathcal{C}_\alpha$.
2. *UMP unbiased (UMPU) test* to a test which is UMP in $\mathcal{C}_{\alpha,U}$.

Example: Let $\overset{X_1,\dots,X_n}{\mathbf{X}} \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$. Consider testing $H_0\colon \mu = \mu_0$ v $H_1\colon \mu \neq \mu_0$ with

$$\boxed{\phi_2(\mathbf{X}) = \mathbf{1}(|T| > t_{n-1,\alpha/2})}, \quad \phi_R(\mathbf{X}) = \mathbf{1}(T > t_{n-1,\alpha}), \quad \phi_L(\mathbf{X}) = \mathbf{1}(T < -t_{n-1,\alpha}),$$

where $T = \sqrt{n}(\bar{X}_n - \mu_0)/S_n$.



$\Theta_0 = \{\mu_0\}$

Discuss unbiasedness, UMP, UMPU property of ϕ_2, ϕ_R, ϕ_L.

One-sided and two-sided sets of hypotheses

Let $\tau = \tau(\theta)$ be one-dimensional.

1. Hypotheses are called *one-sided* if they are of the form

$$H_0\colon \tau \leq \tau_0 \text{ vs } H_1\colon \tau > \tau_0 \quad (\text{or } H_0\colon \tau \geq \tau_0 \text{ vs } H_1\colon \tau < \tau_0)$$

2. Hypotheses are called *two-sided* if they are of the form

$$H_0\colon \tau = \tau_0 \text{ vs } H_1\colon \tau \neq \tau_0$$

$\Theta = \{\theta_0, \theta_1\}$

Neyman-Pearson Lemma

Let $\mathbf{X}$ have the likelihood $\mathcal{L}(\theta; \mathbf{X})$, and suppose we wish to test

$$\boxed{H_0\colon \theta = \theta_0} \text{ versus } H_1\colon \theta = \theta_1.$$

$\mathcal{L}(\theta; \underset{\sim}{X})$

Then for any $k > 0$ the test

$$\phi(\mathbf{X}) = 1 \iff \frac{\mathcal{L}(\theta_0; \mathbf{X})}{\mathcal{L}(\theta_1; \mathbf{X})} < k$$

is at least as powerful as any other test with the same or smaller size.

Moreover, it is the unique such test (ignoring events occurring w/prob 0).

N-P Lemma points us toward ratios of likelihoods in our search for UMP tests.

Exercise: Prove first part of result.

Test: $\phi(\underset{\sim}{x}) = 1$ if $\dfrac{L(\theta_0;\underset{\sim}{x})}{L(\theta_1;\underset{\sim}{x})} < k$

Show that for any other test ϕ^* with same or smaller size, $\phi(\underset{\sim}{x})$ has greater power when $\theta = \theta_1$.

Let $\phi(\underset{\sim}{x})$ have size α.

$\phi^*(\underset{\sim}{x})$ be any other test with size $\alpha^* \le \alpha$.

Want to show power $\phi(\underset{\sim}{x})$ when $\theta = \theta_1$ is at least as large as that of $\phi^*(\underset{\sim}{x})$.

That is, want to show

$$\underbrace{\mathbb{E}_{\theta_1}\phi(\underset{\sim}{x})}_{P_{\theta_1}(\phi(\underset{\sim}{x})=1)} \ge \underbrace{\mathbb{E}_{\theta_1}\phi^*(\underset{\sim}{x})}_{P_{\theta_1}(\phi^*(\underset{\sim}{x})=1)}.$$

Write

$$\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x}) = \begin{cases} \ge 0 & \phi(\underset{\sim}{x}) = 1 \\ \le 0 & \phi(\underset{\sim}{x}) = 0 \end{cases}$$

$\phi(\underset{\sim}{x})$	$\phi^*(\underset{\sim}{x})$
1	1
0	1
1	0
0	0

$$\phi(\underset{\sim}{x}) = 1 \iff \frac{L(\theta_0;\underset{\sim}{x})}{L(\theta_1;\underset{\sim}{x})} < k \iff L(\theta_0;\underset{\sim}{x}) - k\,L(\theta_1;\underset{\sim}{x}) < 0$$

$$0 \ge \underbrace{[\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})]}_{\substack{\ge 0 \text{ if } \phi(\underset{\sim}{x})=1 \\ \le 0 \text{ if } \phi(\underset{\sim}{x})=0}} \underbrace{[L(\theta_0;\underset{\sim}{x}) - k\,L(\theta_1;\underset{\sim}{x})]}_{\substack{<0 \text{ if } \phi(\underset{\sim}{x})=1 \\ >0 \text{ if } \phi(\underset{\sim}{x})=0}}$$

$$0 \geq \int_{\mathcal{X}} [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] [L(\theta_0;\underset{\sim}{x}) - k L(\theta_1;\underset{\sim}{x})] \, d\underset{\sim}{x}$$

$$= \int_{\mathcal{X}} [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] L(\theta_0;\underset{\sim}{x}) \, dx - k \int_{\mathcal{X}} [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] L(\theta_1;\underset{\sim}{x}) \, d\underset{\sim}{x}$$

$$= \underbrace{\mathbb{E}_{\theta_0} \phi(\underset{\sim}{x})} - \mathbb{E}_{\theta_0} \phi^*(\underset{\sim}{x}) - k \left[\mathbb{E}_{\theta_1} \phi(\underset{\sim}{x}) - \mathbb{E}_{\theta_1} \phi^*(\underset{\sim}{x}) \right]$$

$$= \underbrace{\alpha - \alpha^*}_{\geq 0} - k \left[\mathbb{E}_{\theta_1} \phi(\underset{\sim}{x}) - \mathbb{E}_{\theta_1} \phi^*(\underset{\sim}{x}) \right]$$

$$\geq - \underbrace{k}_{\geq 0} \left[\mathbb{E}_{\theta_1} \phi(\underset{\sim}{x}) - \mathbb{E}_{\theta_1} \phi^*(\underset{\sim}{x}) \right]$$

$\Rightarrow$

$$- \left[\mathbb{E}_{\theta_1} \phi(\underset{\sim}{x}) - \mathbb{E}_{\theta_1} \phi^*(\underset{\sim}{x}) \right] \leq 0$$

$$\underbrace{\mathbb{E}_{\theta_1} \phi^*(\underset{\sim}{x})}_{\text{power of } \phi^*(\underset{\sim}{x}) \text{ at } \theta_1} \leq \mathbb{E}_{\theta_1} \phi(\underset{\sim}{x})$$

$\uparrow$ power of $\phi(\underset{\sim}{x})$ at θ_1

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \mathbb{1}(x>0)$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Exponential}(\beta)$, β unknown, and consider

$$H_0\colon \beta = \beta_0 \text{ versus } H_1\colon \beta = \beta_1.$$

Give the form of the most powerful test.

$$L(\beta; \underset{\sim}{X}) = \left(\frac{1}{\beta}\right)^n e^{-\sum_{i=1}^n x_i/\beta} = \left(\frac{1}{\beta}\right) e^{-\frac{n\bar{X}_n}{\beta}}.$$

Reject $H_0: \beta = \beta_0$ if $\dfrac{L(\beta_0; \underset{\sim}{X})}{L(\beta_1; \underset{\sim}{X})} < k$.

i.e. $$\frac{\left(\frac{1}{\beta_0}\right)^n e^{-\frac{n\bar{x}_n}{\beta_0}}}{\left(\frac{1}{\beta_1}\right)^n e^{-n\bar{x}_n/\beta_1}} = \left(\frac{\beta_1}{\beta_0}\right)^n e^{-n\bar{x}_n\left(\frac{1}{\beta_0} - \frac{1}{\beta_1}\right)} < k.$$

$$\iff \quad e^{-n\bar{x}_n\left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right)} < \left(\frac{\beta_0}{\beta_1}\right)^n k$$

$$\iff \quad -n\bar{x}_n\left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right) < n\log\left(\frac{\beta_0}{\beta_1}\right) + \log k$$

$$\bar{x}_n\left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right) > -\log\left(\frac{\beta_0}{\beta_1}\right) - \frac{1}{n}\log k$$

$\iff$

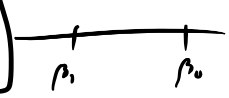
$$\bar{x}_n > \underbrace{\frac{-\log\left(\frac{\beta_0}{\beta_1}\right) - \frac{1}{n}\log k}{\left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right)}}_{k^*} \qquad \left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right) > 0 \iff \beta_0 < \beta_1$$

$$\bar{x}_n < \underbrace{\frac{-\log\left(\frac{\beta_0}{\beta_1}\right) - \frac{1}{n}\log k}{\left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right)}}_{k^*} \qquad \left(\frac{1}{\beta_0}-\frac{1}{\beta_1}\right) < 0 \iff \beta_0 > \beta_1$$

Testing $\boxed{H_0: \beta = \beta_0 \quad \text{vs} \quad H_1: \beta = \beta_1}$

Then we have $\dfrac{L(\beta_0;\underline{X})}{L(\beta_1;\underline{X})} < k$

$$\iff \boxed{\begin{array}{llll} \bar{x}_n > k^* & \text{if} & \beta_0 < \beta_1 \\ \bar{x}_n < k^* & \text{if} & \beta_0 > \beta_1 \end{array}}$$



$$L(\theta;\underset{\sim}{X}) = f(\underset{\sim}{x};\theta) = g(T(\underset{\sim}{x});\theta)\,h(\underset{\sim}{x}) = f_T(T(\underset{\sim}{x});\theta)\,\tilde{h}(\underset{\sim}{x})$$

$$\frac{L(\theta_0;\underset{\sim}{X})}{L(\theta_1;\underset{\sim}{X})} = \frac{f_T(T(\underset{\sim}{x});\theta_0)}{f_T(T(\underset{\sim}{x});\theta_1)}$$

Corollary (UMP test a function of a sufficient statistic)

The UMP test (N-P) of $H_0\colon \theta = \theta_0$ vs $H_1\colon \theta = \theta_1$ is a function of a minimal sufficient statistic, and if $T = T(\mathbf{X})$ is sufficient, the test is equivalent to

$$\phi(\mathbf{X}) = 1 \iff \frac{f_T(T(\mathbf{X});\theta_0)}{f_T(T(\mathbf{X});\theta_1)} < k.$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$, λ unknown, and consider

$$H_0\colon \lambda = \lambda_0 \text{ versus } H_1\colon \lambda = \lambda_1.$$

Give the form of the most powerful test.

$$\frac{L(\lambda_0;\underset{\sim}{X})}{L(\lambda_1;\underset{\sim}{X})} < k \quad , \text{ equivalent to} \quad \frac{f_T(T(\underset{\sim}{x});\theta_0)}{f_T(T(\underset{\sim}{X});\theta_1)} < k$$

$$p_X(x;\lambda) = \frac{e^{-\lambda}\,\lambda^{x}}{x!}$$

$$L(\lambda;\underset{\sim}{x}) = e^{-n\lambda}\,\lambda^{\sum_{i=1}^{n} x_i} \Big/ \prod_{i=1}^{n} x_i!$$

$$\frac{L(\lambda_0;\underset{\sim}{x})}{L(\lambda_1;\underset{\sim}{x})} = \frac{e^{-n\lambda_0}\,\lambda_0^{\sum x_i} \big/ \prod x_i!}{e^{-n\lambda_1}\,\lambda_1^{\sum x_i} \big/ \prod x_i!} = e^{-n(\lambda_0-\lambda_1)}\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum x_i} < k$$

$T = \sum_{i=1}^{n} X_i$ is a suff. stat for λ. $T \sim \text{Poisson}(n\lambda)$.

$$\frac{f_T(T(\underset{\sim}{x});\lambda_0)}{f_T(T(\underset{\sim}{x});\lambda_1)} = \frac{e^{-n\lambda_0}\,\lambda_0^{T(\underset{\sim}{x})} \big/ \cancel{T(\underset{\sim}{x})!}}{e^{-n\lambda_1}\,\lambda_1^{T(\underset{\sim}{x})} \big/ \cancel{T(\underset{\sim}{x})!}} = e^{-n(\lambda_0-\lambda_1)}\left(\frac{\lambda_0}{\lambda_1}\right)^{T(\underset{\sim}{x})} < k$$

If $\dfrac{L(\theta_0;\underset{\sim}{X})}{L(\theta_1;\underset{\sim}{X})}$ is monotone in $T(\underset{\sim}{X})$, a suff stat.

Theorem (Karlin-Rubin: Monotone LR gives UMP for one-sided tests)

Let $\Theta \subset \mathbb{R}$ and let T be a 1-dimensional sufficient statistic with pdf/pmf $f_T(t;\theta)$.

1. *If for $\boxed{\theta_1 > \theta_0}$ the ratio $f_T(t;\theta_0)/f_T(t;\theta_1)$ is non-increasing in t, then*
   - *$T > c$ is the rule of the unique UMP of its size for $H_0\colon \theta \leq \theta_0$ v $H_1\colon \theta > \theta_0$.*
   - *$T < c$ is the rule of the unique UMP of its size for $H_0\colon \theta \geq \theta_0$ v $H_1\colon \theta < \theta_0$.*
2. *If for $\theta_1 > \theta_0$ the ratio $f_T(t;\theta_0)/f_T(t;\theta_1)$ is non-decreasing in t, then*
   - *$T < c$ is the rule of the unique UMP of its size for $H_0\colon \theta \leq \theta_0$ v $H_1\colon \theta > \theta_0$.*
   - *$T > c$ is the rule of the unique UMP of its size for $H_0\colon \theta \geq \theta_0$ v $H_1\colon \theta < \theta_0$.*

We say that T has a *monotone likelihood ratio (MLR)* in the above cases.

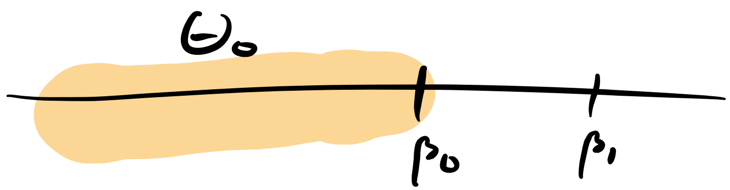
Easy MLR check: Just see if $\frac{\mathcal{L}(\theta_0;\mathbf{X})}{\mathcal{L}(\theta_1;\mathbf{X})}$ is monotone in a suff. stat when $\theta_0 \neq \theta_1$.

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Exponential(β). Give UMP size-α test for

1. H_0: $\beta \leq \beta_0$ vs H_1: $\beta > \beta_0$.
2. H_0: $\beta \geq \beta_0$ vs H_1: $\beta < \beta_0$.

based on $\bar{X}_n$.

②



For $\beta_1 > \beta_0$,

$$\frac{L(\beta_0; \underset{\sim}{X})}{L(\beta_1; \underset{\sim}{X})} = \underbrace{\left(\frac{\beta_1}{\beta_0}\right)^n e^{-n\bar{x}_n \boxed{\underbrace{\left(\frac{1}{\beta_0} - \frac{1}{\beta_1}\right)}_{>0}}}}_{\text{monotone decreasing in } \bar{x}_n \text{, so reject when } \bar{x}_n \text{ is large.}} < k$$

$\iff \bar{x}_n > k^*$

Reject $H_0: \beta \le \beta_0$ when $\bar{X}_n > k^*$.

calibrate k^* to give test size α.

$$\gamma(\beta) = P_\beta(\bar{X}_n > k^*)$$

$X_1, \dots, X_n \overset{iid}{\sim} \text{Exponential}(\beta)$

$$= 1 - F_{\text{Gamma}(n, \frac{\beta}{n})}(k^*)$$

$\bar{X}_n \sim \text{Gamma}(n, \frac{\beta}{n})$

$$\text{size} = \sup_{\beta \le \beta_0} \gamma(\beta) = \sup_{\beta \le \beta_0} P_\beta(\bar{X}_n > k^*)$$

$$= P_{\beta_0}(\bar{X}_n > k^*)$$

$$= 1 - F_{\text{Gamma}(n, \frac{\beta_0}{n})}(k^*)$$

$$\overset{\text{set}}{=} \alpha$$

$k^* =$ upper α-qtl of Gamma$(n, \frac{\beta_0}{n})$

$\left(G(n, \frac{\beta_0}{n}, \alpha)\right)$

(2)

If we flip the signs, reject for small $\bar{X}_n$.

$$f(x;\mu) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$L(\mu;\underset{\sim}{X}) = (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right]$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, σ^2 known. Give UMP size-α test for

1. H_0: $\mu \leq \mu_0$ vs H_1: $\mu > \mu_0$.
2. $\boxed{H_0\text{: } \mu \geq \mu_0 \text{ vs } H_1\text{: } \mu < \mu_0}$

based on $\mathbf{X}_n$.

match hypotheses

set $\boxed{\mu_1 < \mu_0}$

$$\frac{L(\mu_0;\underset{\sim}{X})}{L(\mu_1;\underset{\sim}{X})} = \frac{\cancel{(2\pi)^{-n/2}}\, \cancel{(\sigma^2)^{-n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu_0)^2\right]}{\cancel{(2\pi)^{-n/2}}\, \cancel{(\sigma^2)^{-n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu_1)^2\right]}$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} (x_i-\mu_0)^2 - \sum_{i=1}^{n} (x_i-\mu_1)^2\right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n x_i \mu_0 + n\mu_0^2 - \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n x_i \mu_1 - n\mu_1^2 \right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(n\mu_0^2 - n\mu_1^2 - n\bar{x}_n(\mu_0 - \mu_1)\right)\right]$$

$\mu_1 < \mu_0$

$$= \exp\left[\frac{\bar{x}_n(\mu_0-\mu_1)}{2\sigma^2}\right] \exp\left[-\frac{n}{2\sigma^2}(\mu_0^2 - \mu_1^2)\right]$$

$$< k$$

$$\iff \bar{x}_n < k^*$$

Reject $H_0: \mu \geq \mu_0$ when $\bar{x}_n < k^*$.

$H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ reject H_0 when $\frac{L(\theta_0; \underset{\sim}{X})}{L(\theta_1; \underset{\sim}{X})}$ big? small?

Theorem (Karlin-Rubin: Monotone LR gives UMP for one-sided tests)

Let $\Theta \subset \mathbb{R}$ and let T be a 1-dimensional sufficient statistic with pdf/pmf $f_T(t;\theta)$.

1. *If testing $H_0\colon \theta \leq \theta_0$ vs $H_1\colon \theta > \theta_0$, choose any $\theta_1 > \theta_0 \ldots$*
2. *If testing $H_0\colon \theta \geq \theta_0$ vs $H_1\colon \theta < \theta_0$, choose any $\theta_1 < \theta_0 \ldots$*

In either case the unique UMP test is found as:

$$\boxed{\frac{L(\theta_0; \underset{\sim}{X})}{L(\theta_1; \underset{\sim}{X})}} = \frac{f_T(T(\underset{\sim}{X}); \theta_0)}{f_T(T(\underset{\sim}{X}); \theta_1)}$$

If $\boxed{\dfrac{f_T(t;\theta_0)}{f_T(t;\theta_1)}}$ is non-increasing (non-decr) in t, reject H_0 when $T > c$ ($T < c$).

If $\dfrac{f_T(t;\theta_0)}{f_T(t;\theta_1)}$ is monotone in t, we say T has a *monotone likelihood ratio (MLR)*.

Simpler: Just check if $\dfrac{\mathcal{L}(\theta_0; \mathbf{X})}{\mathcal{L}(\theta_1; \mathbf{X})}$ is monotone in a suff. stat when $\theta_0 \neq \theta_1$.

$$L(\mu; \underset{\sim}{X}) = (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2} \right]$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\boxed{\sigma^2 \text{ known}}$. Give UMP $\boxed{\text{size-}\alpha}$ test for

1. $H_0\colon \mu \le \mu_0$ vs $H_1\colon \mu > \mu_0$.
2. $H_0\colon \mu \ge \mu_0$ vs $H_1\colon \mu < \mu_0$. $\longrightarrow$ Reject when $\bar{X}_n < c$,

$$\boxed{\bar{X}_n < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}}$$

(1) set $\mu_1 > \mu_0$, write

(i)
$$\frac{L(\mu_0; \underset{\sim}{X})}{L(\mu_1; \underset{\sim}{X})} = \frac{(2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\sigma^2} \right]}{(2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n (x_i - \mu_1)^2}{2\sigma^2} \right]}$$

$$= \exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu_1)^2}{2\sigma^2}\right]$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left[\cancel{\sum x_i^2} - 2n\bar{x}_n\mu_0 + n\mu_0^2 - \cancel{\sum x_i^2} + 2n\bar{x}_n\mu_1 - n\mu_1^2\right]\right]$$

$$\boxed{= \exp\left[-\frac{1}{2\sigma^2}\left[n(\mu_0^2-\mu_1^2) + 2n\bar{x}_n\underbrace{(\mu_1-\mu_0)}_{>0}\right]\right]}$$

Monotone in $\bar{x}$?

decreasing in $\bar{x}_n$.

Recall: $\bar{x}_n$ is suff for μ.

So reject $H_0: \mu \le \mu_0$ when $\bar{x}_n > c$.

(ii) Find c such that size of test is α.

$$\beta(\mu) = P_\mu(\bar{x}_n > c).$$

$Z \sim N(0,1)$

$$\text{size} = \sup_{\mu \le \mu_0} \beta(\mu)$$

$$= \beta(\mu_0)$$

$$= P_{\mu_0}(\bar{x}_n > c)$$

$$= P_{\mu_0}\left(\frac{\sqrt{n}(\bar{x}_n-\mu_0)}{\sigma} > \frac{\sqrt{n}(c-\mu_0)}{\sigma}\right)$$

$$= P\left(Z > \frac{\sqrt{n}(c-\mu_0)}{\sigma}\right) = \alpha$$

$$\frac{\sqrt{n}(c-\mu_0)}{\sigma} = z_\alpha$$

$\Longleftrightarrow$

$$c = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\boxed{\text{Rej } H_0 \text{ when } \bar{x}_n > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}.}$$

$$L(\theta; \underset{\sim}{X}) = \left(\frac{1}{\theta}\right)^n \mathbb{1}(X_{(n)} < \theta)\, \mathbb{1}(X_{(1)} > 0)$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Uniform(θ). Give the UMP $\boxed{\text{size-}\alpha}$ test for

1. $H_0\colon \theta \leq \theta_0$ vs $H_1\colon \theta > \theta_0$.
2. $H_0\colon \theta \geq \theta_0$ vs $H_1\colon \theta < \theta_0$.

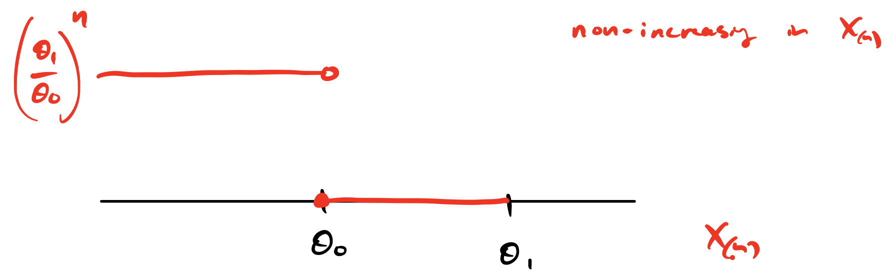
(1) set $\theta_1 > \theta_0$. Then write

(i) $$\frac{L(\theta_0; \underset{\sim}{X})}{L(\theta_1; \underset{\sim}{X})} = \frac{\left(\frac{1}{\theta_0}\right)^n \mathbb{1}(X_{(n)} < \theta_0)}{\left(\frac{1}{\theta_1}\right)^n \mathbb{1}(X_{(n)} < \theta_1)} = \left(\frac{\theta_1}{\theta_0}\right)^n \frac{\mathbb{1}(X_{(n)} < \theta_0)}{\mathbb{1}(X_{(n)} < \theta_1)}$$

$\theta_0(1-\alpha)^{\frac{1}{n}} < X_{(n)} \Rightarrow$ reject H_0

$T(\underset{\sim}{X}) = X_{(n)}$.

$$\frac{L(\theta_0;\underline{x})}{L(\theta_1;\underline{x})}$$



Reject $H_0: \theta \le \theta_0$ when $X_{(n)} > c$.

(ii) $\quad \gamma(\theta) = P_\theta\left(X_{(n)} > c\right)$ $\qquad F_{X_{(n)}}(x) = \left[F_X(x)\right]^n$

$$= \begin{cases} 0, & x \le 0 \\ \left(\frac{x}{\theta}\right)^n & 0 < x < \theta \\ 1 & x \ge \theta \end{cases}$$

$$= 1 - P_\theta\left(X_{(n)} \le c\right)$$

$$= \begin{cases} 1 - 0 & \text{if } c \le 0 \\ 1 - \left(\frac{c}{\theta}\right)^n & \text{if } 0 < c < \theta \\ 1 - 1 & \text{if } c \ge \theta \end{cases}$$

$$= \begin{cases} 1 & \text{if } c \le 0 \\ 1 - \left(\frac{c}{\theta}\right)^n & \text{if } 0 < c < \theta \\ 0 & \text{if } c \ge \theta \end{cases}$$

If $c \in (0, \theta)$

$$\text{size} = \sup_{\theta \le \theta_0} \gamma(\theta)$$

$$= \gamma(\theta_0)$$

$$= 1 - \left(\frac{c}{\theta_0}\right)^n = \alpha$$

$$= \alpha$$

$$\iff 1 - \alpha = \left(\frac{c}{\theta_0}\right)^n$$

$$\iff (1-\alpha)^{\frac{1}{n}} = \frac{c}{\theta_0}$$

$$\iff \theta_0 (1-\alpha)^{\frac{1}{n}} = c$$

Reject $H_0: \theta \le \theta_0$ if $X_{(n)} > \theta_0 (1-\alpha)^{1/n}$

$$p_X(x) = p(1-p)^{x-1}\,\mathbb{1}(x=\{1,2,\dots\})$$

$$L(p;\underset{\sim}{X}) = \prod_{i=1}^{n} p(1-p)^{X_i-1} = p^n(1-p)^{\sum X_i - n} = \left(\frac{p}{1-p}\right)^n (1-p)^{\sum_{i=1}^n X_i}$$

$$T(\underset{\sim}{X}) = \sum_{i=1}^n X_i$$

Exercise: Let $X_1, \dots, X_n \overset{\text{ind}}{\sim}$ Geometric(p). Give a level-α test that is UMP for

1. H_0: $p \leq p_0$ vs H_1: $p > p_0$.
2. H_0: $p \geq p_0$ vs H_1: $p < p_0$.

(2) set $p_1 < p_0$. Then write

(i)
$$\frac{L(p_0;\underset{\sim}{X})}{L(p_1;\underset{\sim}{X})} = \frac{\left(\frac{p_0}{1-p_0}\right)^n (1-p_0)^{\sum_{i=1}^n X_i}}{\left(\frac{p_1}{1-p_1}\right)^n (1-p_1)^{\sum_{i=1}^n X_i}} = \frac{\left(\frac{p_0}{1-p_0}\right)^n}{\left(\frac{p_1}{1-p_1}\right)^n} \underbrace{\underbrace{\left(\frac{1-p_0}{1-p_1}\right)}_{<1}{}^{\sum_{i=1}^n X_i}}_{\searrow \text{ in } \sum_{i=1}^n X_i}$$

So reject $\boxed{H_0: p \geq p_0 \quad \text{when} \quad \underline{\underline{\sum_{i=1}^n X_i}} > c}$.

(ii) $\quad \beta(p) = P_p\left(\sum_{i=1}^n X_i > c\right)$

$$\text{size} = \sup_{p \geq p_0} \beta(p)$$

Negative Binomial (n, p_0)

$$= P_{p_0}\left(\sum_{i=1}^n X_i > c\right)$$

trials until n successes

Take c = upper α quantile of Neg Binom (n, p_0).

This gives size $\leq \alpha$

The Karlin-Rubin result says we can find UMP tests for:

- One-sided tests of hypotheses.
- When the parameter is one-dimensional.
- When there is a one-dimensional sufficient statistic.

This does not cover all situations!

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