# STAT 713 sp 2023 Lec 12 slides

## Likelihood ratio tests

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

## Likelihood ratio

For $\mathbf{X}$ having likelihood $\mathcal{L}(\theta; \mathbf{X})$ and for some hypotheses

$$H_0\colon \theta \in \Theta_0 \text{ versus } H_1\colon \theta \in \Theta_1,$$

the *likelihood ratio (LR)* is defined as

$$\mathrm{LR}(\mathbf{X}) = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta; \mathbf{X})}{\sup_{\theta \in \Theta} \mathcal{L}(\theta; \mathbf{X})}.$$

The likelihood ratio must take values in the interval ________.

A ________ (larger/smaller) likelihood ratio casts ________ (more/less) doubt on $H_0$.

## Likelihood ratio test

A *likelihood ratio test (LRT)* is a test which rejects $H_0$ if $\text{LR}(\mathbf{X}) < k$ for some $k$.

A _______ value of $k$ gives the test _______ power and _______ size.

The critical value $k$ can be chosen to give the test a desired size.

Karlin-Rubin: LRTs are UMP for one-sided tests when there is a one-dimensional parameter and a one-dimensional sufficient statistic.

**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\sigma^2$ known. For

$$H_0\colon \mu = \mu_0 \text{ versus } H_1\colon \mu \neq \mu_0$$

1. Give the LR.
2. Calibrate the LRT to have size $\alpha$ for any $\alpha \in (0, 1)$.

## Finding the likelihood ratio

We can write the likelihood ratio as

$$\mathrm{LR}(\mathbf{X}) = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta; \mathbf{X})}{\sup_{\theta \in \Theta} \mathcal{L}(\theta; \mathbf{X})} = \frac{\mathcal{L}(\hat{\theta}_0; \mathbf{X})}{\mathcal{L}(\hat{\theta}; \mathbf{X})},$$

where

$$\hat{\theta}_0 = \operatorname{argmax}_{\theta \in \Theta_0} \ell(\theta; \mathbf{X})$$
$$\hat{\theta} = \operatorname{argmax}_{\theta \in \Theta} \ell(\theta; \mathbf{X})$$

So we just need to find these and plug them in:

- $\hat{\theta}_0$ is a *restricted maximum likelihood estimator*; best estimator in null space.
- $\hat{\theta}$ is the MLE.

Note: If the null is a simple hypothesis, i.e. $H_0$: $\theta = \theta_0$, then $\hat{\theta}_0 = \theta_0$.

Also: If $\hat{\theta} \in \Theta_0$, then $\hat{\theta} = \hat{\theta}_0$, so that $\mathrm{LR}(\mathbf{X}) = 1$.

**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\sigma^2$ known. Consider testing

$$H_0\colon \mu \leq \mu_0 \text{ versus } H_1\colon \mu > \mu_0.$$

1. Give the LR.
2. Calibrate the LRT to have size $\alpha$ for any $\alpha \in (0, 1)$.

**Exercise:** Let $Y \sim \text{Geometric}(p)$ distribution, $p \in [0, 1]$ unknown. Consider testing

$$H_0\colon p \leq p_0 \text{ versus } H_1\colon p > p_0.$$

1. Give the LR.
2. Give a level-$\alpha$ likelihood ratio test.

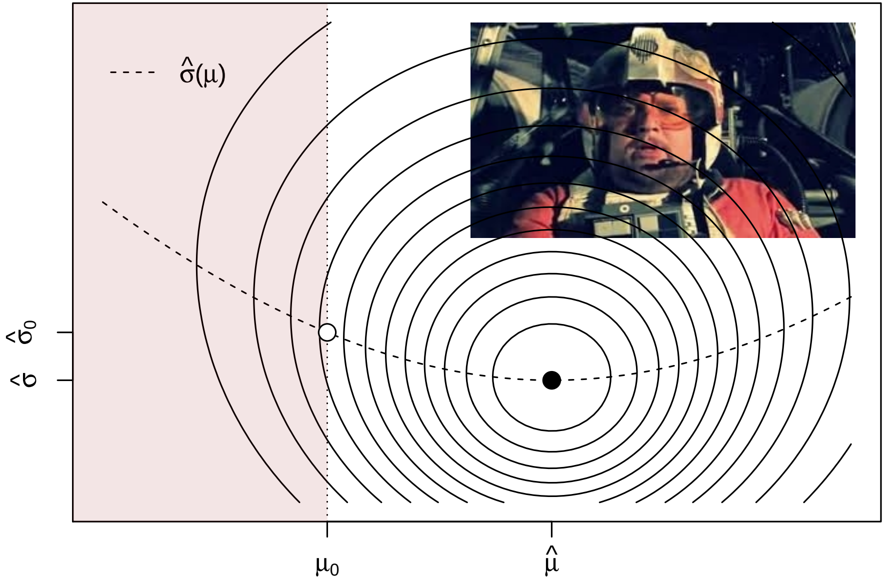
1. LRTs for Normal mean and variance

2. The asymptotic likelihood ratio test

**Exercise:** Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\mu, \sigma^2$ unknown, Consider testing

$$H_0\colon \mu \leq \mu_0 \text{ versus } H_1\colon \mu > \mu_0.$$

Show that the size-$\alpha$ LRT has rejection rule $\sqrt{n}(\bar{X}_n - \mu_0)/S_n > t_{n-1,\alpha}$.


$\hat{\sigma}(\mu)$
$\hat{\sigma}_0$
$\hat{\sigma}$
$\mu_0$
$\hat{\mu}$

**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\mu, \sigma^2$ unknown. Consider testing

$$H_0\colon \sigma^2 = \sigma_0^2 \text{ versus } H_1\colon \sigma^2 \neq \sigma_0^2.$$

1. Show that the LRT rejection rule can be written

$$(n-1)S_n^2/\sigma_0^2 < c_1 \quad \text{or} \quad (n-1)S_n^2/\sigma_0^2 > c_2.$$

2. Consider choosing $c_1$ and $c_2$ so that the LRT has size $\alpha$.

## Wilk's Theorem

Let $\mathbf{X}_n$ be a random sample with likelihood $\mathcal{L}(\theta; \mathbf{X}_n)$, $\theta \in \Theta$, and consider

$$H_0\colon \tau(\theta) = \tau_0 \text{ versus } H_1\colon \tau(\theta) \neq \tau_0,$$

where $\dim(\Theta) = d$ and $\Theta_0 = \{\theta \in \Theta : \tau(\theta) = \tau_0\}$, with $\dim(\Theta_0) = d_0 < d$.

Then under $H_0$, we have

$$-2 \log \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta; \mathbf{X}_n)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta; \mathbf{X}_n)} \xrightarrow{\text{D}} \chi^2_{d-d_0}$$

as $n \to \infty$, provided the conditions for asymptotic Normality of MLEs hold.

**Exercise:** Go through heuristics of proof for simple case $H_0\colon \theta = \theta_0$.

Wilk's Theorem allows us to define the following test:

### Definition (Asymptotic likelihood ratio test)

Under the settings of Wilk's theorem, the test with rejection rule

$$\text{Reject } H_0 \text{ iff } -2\log\frac{\sup_{\theta\in\Theta_0}\mathcal{L}(\theta;\mathbf{X}_n)}{\sup_{\theta\in\Theta}\mathcal{L}(\theta;\mathbf{X}_n)} > \chi^2_{d-d_0,\alpha}.$$

has size approximately $\alpha$ for large $n$. It is called the *asymptotic* LRT.

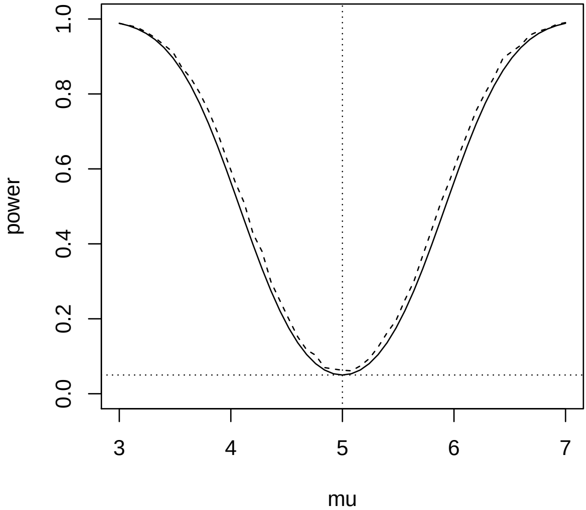
The asymptotic LRT is only for two-sided hypotheses.

- $d_0 = \dim(\Theta_0)$ is the number of parameters left unspecified by $H_0$.
- $d = \dim(\Theta)$ is the total number of unknown parameters.

**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, $\mu$ and $\sigma^2$ unknown, and consider

$$H_0\colon \mu = \mu_0 \text{ versus } H_1\colon \mu \neq \mu_0.$$

1. Find the size-$\alpha$ asymptotic LRT.
2. Compare via simulation the power curves of the asymptotic LRT and the (non-asymptotic) LRT. Use the settings $n = 20$ and $\alpha = 0.05$.


power
0.0
0.2
0.4
0.6
0.8
1.0
3
4
5
6
7
mu

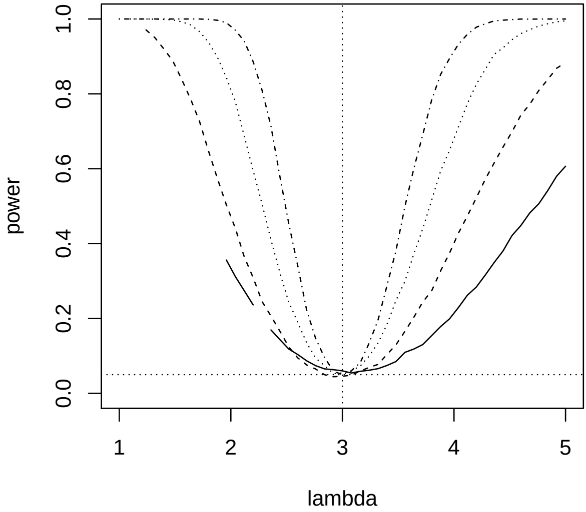
**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$, $\lambda$ unknown, and consider testing

$$H_0\colon \lambda = \lambda_0 \text{ versus } H_1\colon \lambda \neq \lambda_0.$$

1. Give the decision rule of the asymptotic LRT.
2. Let $\lambda_0 = 3$ and run a simulation to get power curves for the test under the sample sizes $n = 5, 10, 20, 40$ using $\alpha = 0.05$.
3. Find the $p$-value of the asymptotic likelihood ratio test of

$$H_0\colon \lambda = 3 \text{ versus } H_1\colon \lambda \neq 3$$

associated with a sample of size $n = 25$ with sample mean equal to $3.5$.


power
1.0
0.8
0.6
0.4
0.2
0.0
1
2
3
4
5
lambda

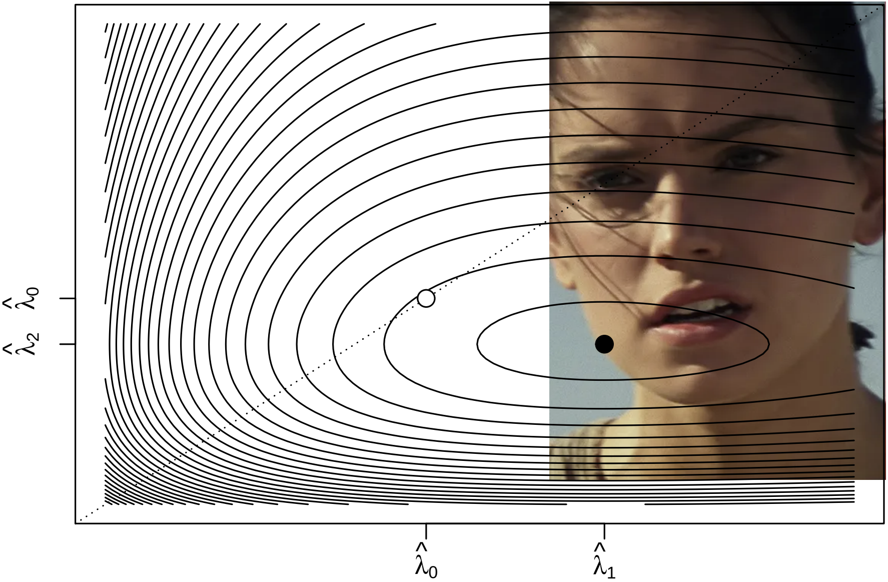
**Exercise:** Let $X_{k1}, \dots, X_{kn_k} \overset{\text{ind}}{\sim} \text{Exponential}(\lambda_k)$, $k = 1, 2$ and consider

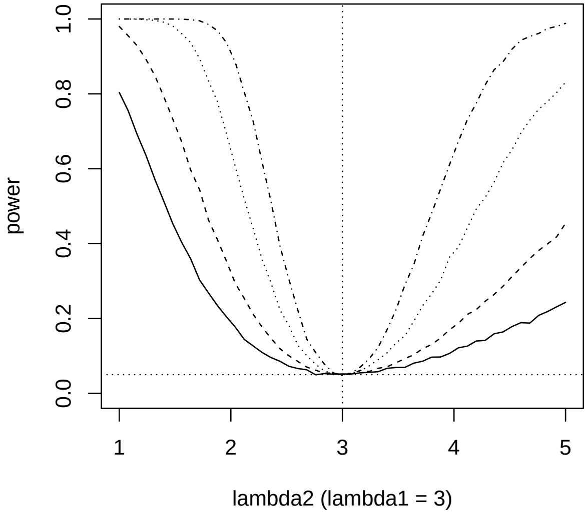
$$H_0\colon \lambda_1 = \lambda_2 \text{ versus } H_1\colon \lambda_1 \neq \lambda_2.$$

- Give the decision rule of the asymptotic LRT.
- Plot power curves of the test with $\alpha = 0.05$ under

  $$(n_1, n_2) = (10, 20), (20, 40), (50, 100), (100, 200)$$

  when $\lambda_1 = 3$ with $\lambda_2$ varying from $1$ to $5$.
- Find the $p$-value of the asymptotic LRT associated with observing $\bar{X}_1 = 4.1$ and $\bar{X}_2 = 3.2$ when $n_1 = 30$ and $n_2 = 34$.


$\hat{\lambda}_0$
$\hat{\lambda}_2$
$\hat{\lambda}_0$
$\hat{\lambda}_1$


power
0.0
0.2
0.4
0.6
0.8
1.0
1
2
3
4
5
lambda2 (lambda1 = 3)

**Exercise:** For some $K \geq 1$ and $M \geq 1$, suppose we observe indep. random vectors

$$Y_m \sim \text{Multinomial}(n_m, p_{m1}, \dots, p_{mK}), \quad \text{for } m = 1, \dots, M.$$

Derive the asymptotic size-$\alpha$ likelihood ratio test for

$$H_0 : p_{ik} = p_{jk} \text{ for all } i \neq j \text{ for each } k = 1, \dots, K.$$