STAT 713 hw 1

Sufficiency, minimality, ancillarity, completeness

Do problems 6.7, 6.9 (a)(b)(c), 6.13, 6.14, 6.15, 6.20, 6.21, 6.22 from CB. In addition:

1. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Uniform}(\theta - 1, \theta + 1)$, $\theta \in \mathbb{R}$.
   (a) Show that $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimum sufficient statistic for θ.

   We have $f(x; \theta) = (1/2)\mathbf{1}(\theta - 1 < x < \theta + 1)$, so the joint pdf of $X_1, \ldots, X_n$ is given by

   $$f(\mathbf{x}; \theta) = \prod_{i=1}^{n} (1/2) \cdot \mathbf{1}(\theta - 1 < x_i < \theta + 1) = (1/2)^n \cdot \mathbf{1}(\theta - 1 < x_{(1)}) \cdot \mathbf{1}(x_{(n)} < \theta + 1).$$

   For two samples $\mathbf{x}$ and we have

   $$\frac{f(\mathbf{x}; \theta)}{f(; \theta)} = \frac{(1/2)^n \cdot \mathbf{1}(\theta - 1 < x_{(1)}) \cdot \mathbf{1}(x_{(n)} < \theta + 1)}{(1/2)^n \cdot \mathbf{1}(\theta - 1 < y_{(1)}) \cdot \mathbf{1}(y_{(n)} < \theta + 1)},$$

   of which the numerator and denominator are nonzero for the same values of θ if and only if $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$, over which same values of θ the numerator and denominator are constant if and only if $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$. Therefore $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic.

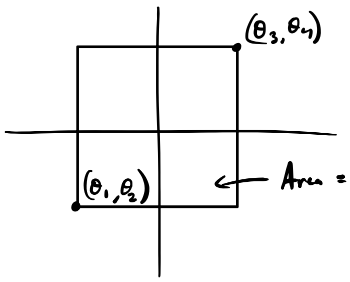
   (b) Show that $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not complete.

   For every $\theta \in \mathbb{R}$, the pdf $f(x; \theta) = (1/2)\mathbf{1}(\theta - 1 < x < \theta + 1)$ can be written as $f(x; \theta) = f_Z(x - \theta)$, where $f_Z(z) = (1/2)\mathbf{1}(-1 < z < 1)$. Therefore $f(x; \theta)$ belongs to a scale family. In consequence, the statistic $R(\mathbf{X}) = X_{(n)} - X_{(1)}$ is ancillary; since we can define an ancillary statistic as a function of $T(\mathbf{X})$, the latter is not a complete statistic.

2. (Optional) Additional problems from CB: 6.9 (d)(e), 6.23.

Problems 6.7, 6.9 (a)(b)(c), 6.13, 6.14, 6.15, 6.20, 6.21, 6.22 from CB.

$\boxed{6.7}$ Let $f(x, y; \theta_1, \theta_2, \theta_3, \theta_4) = \dfrac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} \mathbb{1}(\theta_1 < x < \theta_3,\ \theta_2 < y < \theta_4)$.



$\theta_1 < \theta_3,\ \theta_2 < \theta_4$

← Area $= (\theta_3-\theta_1)(\theta_4-\theta_2)$

The joint pdf of $(X_1, Y_1), \dots, (X_n, Y_n) \overset{ind}{\sim} f(x, y; \theta_1, \theta_2, \theta_3, \theta_4)$ is

$$f(\underset{\sim}{x}, \underset{\sim}{y}; \theta_1, \theta_2, \theta_3, \theta_4) = \prod_{i=1}^{n} \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} \mathbb{1}(\theta_1 < x < \theta_3,\ \theta_2 < y < \theta_4).$$

$$= \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} \mathbb{1}(\theta_1 < x_{(1)}) \mathbb{1}(x_{(n)} < \theta_3) \mathbb{1}(\theta_2 < y_{(1)}) \mathbb{1}(y_{(n)} < \theta_4),$$

so, by the factorization theorem, $T(\underset{\sim}{X}, \underset{\sim}{Y}) = (X_{(1)}, X_{(n)}, Y_{(1)}, Y_{(n)})$ is a sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$.

$\boxed{6.9}$ (a)

$$f(\underset{\sim}{x}; \theta) = (2\pi)^{-n/2} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right]$$

$$= (2\pi)^{-n/2} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - 2\theta \sum x_i + n\theta^2\right)\right]$$

$$= \underbrace{e^{\theta n \bar{x}_n} e^{-\frac{n\theta^2}{2}}}_{g(\bar{x}_n; \theta)} \underbrace{(2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}_{h(\underset{\sim}{x})}.$$

The factorization theorem gives that $T(\underset{\sim}{X}) = \bar{X}_n$ is a suff. stat. for θ.

To check minimality, write

$$\frac{f(\underset{\sim}{x}; \theta)}{f(\underset{\sim}{y}; \theta)} = \frac{e^{\theta n \bar{x}_n} e^{-n\theta^2/2}}{e^{\theta n \bar{y}_n} e^{-n\theta^2/2}} \cdot \frac{(2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}{(2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n} y_i^2}} = \frac{e^{\theta n \bar{x}_n}}{e^{\theta n \bar{y}_n}} \frac{e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}{e^{-\frac{1}{2}\sum_{i=1}^{n} y_i^2}},$$

and note that it is free of θ if and only if $\bar{x}_n = \bar{y}_n$.

Therefore $T(\underset{\sim}{X}) = \bar{X}_n$ is a minimal sufficient statistic for θ.

(b) $$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} \mathbb{1}(x_i > \theta)$$
$$= e^{-\sum_{i=1}^{n} x_i} e^{-n\theta} \prod_{i=1}^{n} \mathbb{1}(x_i > \theta)$$
$$= \underbrace{e^{-n\bar{x}_n}}_{h(\underset{\sim}{x})} \underbrace{e^{-n\theta} \mathbb{1}(x_{(1)} > \theta)}_{g(x_{(1)};\theta)} .$$

The factorization theorem gives that $T(\underset{\sim}{X}) = X_{(1)}$ is a suff. stat. for θ.

Now write

$$\frac{f(\underset{\sim}{x};\theta)}{f(\underset{\sim}{y};\theta)} = \frac{e^{-n\bar{x}_n} e^{-n\theta} \mathbb{1}(x_{(1)} > \theta)}{e^{-n\bar{y}_n} e^{-n\theta} \mathbb{1}(y_{(1)} > \theta)} = \frac{e^{-n\bar{x}_n}}{e^{-n\bar{y}_n}} \frac{\mathbb{1}(x_{(1)} > \theta)}{\mathbb{1}(y_{(1)} > \theta)} .$$

If $x_{(1)} = y_{(1)}$, then the numerator and the denominator have the same support and the ratio is constant in θ when the denominator is positive (and vice versa).

Therefore $T(\underset{\sim}{X}) = X_{(1)}$ is a minimal sufficient statistic.

(c) $$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \frac{e^{-(x_i-\theta)}}{\left[1 + e^{-(x_i-\theta)}\right]^2} = \underbrace{\frac{e^{-n(\bar{x}_n-\theta)}}{\prod_{i=1}^{n}\left[1 + e^{-(x_{(i)}-\theta)}\right]^2}}_{g(x_{(1)},\dots,x_{(n)};\theta)} \cdot \underbrace{1}_{h(\underset{\sim}{x})} .$$

The factorization theorem gives that $T(\underset{\sim}{X}) = (X_{(1)},\dots,X_{(n)})$.

Now write

$$\frac{f(\underset{\sim}{x};\theta)}{f(\underset{\sim}{y};\theta)} = \frac{e^{-n(\bar{x}_n-\theta)} \Big/ \prod_{i=1}^{n}\left[1 + e^{-(x_{(i)}-\theta)}\right]^2}{e^{-n(\bar{y}_n-\theta)} \Big/ \prod_{i=1}^{n}\left[1 + e^{-(y_{(i)}-\theta)}\right]^2} .$$

The above is free of θ iff $(x_{(1)},\dots,x_{(n)}) = (y_{(1)},\dots,y_{(n)})$.

Therefore $T(\underset{\sim}{X}) = (X_{(1)},\dots,X_{(n)})$ is a minimal sufficient statistic.

Why do we have to "switch" to order statistics?

Why do we not just take the whole sample? Because

if $$\frac{e^{-n(\bar{x}_n-\theta)} \Big/ \prod_{i=1}^{n} \left[1+e^{-(x_i-\theta)}\right]^2}{e^{-n(\bar{y}_n-\theta)} \Big/ \prod_{i=1}^{n} \left[1+e^{-(y_i-\theta)}\right]^2} \text{ free of } \theta$$

then it is free of θ for any re-ordering of $(X_1,\dots,X_n)$ and $(Y_1,\dots,Y_n)$. Therefore the condition only implies that $(X_1,\dots,X_n)$ and $(Y_1,\dots,Y_n)$ have the same <u>unique</u> values (are the same when sorted).

$\boxed{6.13}$ Let $X_1, X_2 \overset{ind}{\sim} f(x;\alpha) = \alpha x^{\alpha-1} e^{-x^\alpha} \mathbb{1}(x>0)$ for $\alpha>0$.

Set $Y = \log X_1$. Then $Y \in \mathbb{R}$. Moreover

$$y = \log x = g(x) \iff x = e^y = g^{-1}(y), \quad \frac{d}{dy} g^{-1}(y) = e^y,$$

so

$$f_Y(y;\alpha) = \alpha e^{y(\alpha-1)} e^{-e^{y\alpha}} e^y = \alpha e^{y\alpha} e^{-e^{y\alpha}}.$$

Note that

$$f_Y(y;\alpha) = \frac{1}{\alpha^{-1}} f_Z\left(y/\alpha^{-1}\right), \quad \text{where } f_Z(z) = e^z e^{-e^z} \quad \forall\, \alpha>0,$$

so $f_Y(y;\alpha)$ belongs to a scale family.

Thus we can write

$$Y = \alpha^{-1} Z, \quad \text{where } Z \sim f_Z(z) = e^z e^{-e^z}.$$

Now we put $\log X_1 = \alpha^{-1} Z_1$ and $\log X_2 = \alpha^{-1} Z_2$, where $Z_1, Z_2 \overset{ind}{\sim} f_Z$.

This gives

$$\frac{\log X_1}{\log X_2} = \frac{\alpha^{-1} Z_1}{\alpha^{-1} Z_2} = \frac{Z_1}{Z_2},$$

which has a distribution not depending on α, so it is ancillary.

$\boxed{6.14}$ Let $X_1, \dots, X_n$ be a rs from a location family. Show that $M - \bar{X}_n$ is ancilly, where M is the median.

Since $X_1, \dots, X_n$ come from a location family we may write

$$X_i = Z_i + \mu, \qquad i = 1, \dots, n$$

for some location parameter $\mu \in \mathbb{R}$, where the distribution of Z_i does not depend on μ.

Then

$$M - \bar{X}_n = \begin{cases} X_{(\frac{n}{2})} - \frac{1}{n}\sum_{i=1}^n X_i, & n \text{ even} \\ \frac{X_{(\lfloor \frac{n}{2} \rfloor)} + X_{(\lceil \frac{n}{2} \rceil)}}{2} - \frac{1}{n}\sum_{i=1}^n X_i, & n \text{ odd} \end{cases}$$

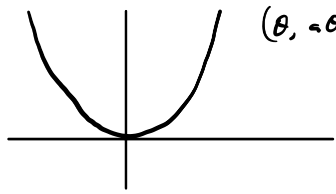
$$= \begin{cases} Z_{(\frac{n}{2})} + \mu - \frac{1}{n}\sum_{i=1}^n (Z_i + \mu), & n \text{ even} \\ \frac{Z_{(\lfloor \frac{n}{2} \rfloor)} + \mu + Z_{(\lceil \frac{n}{2} \rceil)} + \mu}{2} - \frac{1}{n}\sum_{i=1}^n (Z_i + \mu), & n \text{ odd} \end{cases}$$

$$= \begin{cases} Z_{(\frac{n}{2})} - \frac{1}{n}\sum_{i=1}^n Z_i, & n \text{ even} \\ \frac{Z_{(\lfloor \frac{n}{2} \rfloor)} + Z_{(\lceil \frac{n}{2} \rceil)}}{2} - \frac{1}{n}\sum_{i=1}^n Z_i, & n \text{ odd,} \end{cases}$$

which has a distribution free of μ and is therefore ancillary.

$\boxed{6.15}$ Let $X_1, \dots, X_n \overset{ind}{\sim} \text{Normal}(\mu = \theta, \sigma^2 = a\theta^2)$, $a > 0$ known, $\theta \in \mathbb{R}$.

(a) The parameter space for $(\mu, \sigma^2) = (\theta, a\theta^2)$ is a curve:



$(\theta, a\theta^2)$

(b) The joint pdf of $X_1, \dots, X_n$ admits the factorization

$$f(\underset{\sim}{x}; \theta) = (2\pi)^{-n/2} (a\theta^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n (x_i - \theta)^2}{2a\theta^2} \right]$$

$$= (2\pi)^{-n/2} (a\theta^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n ((x_i - \bar{x}_n) + (\bar{x}_n - \theta))^2}{2a\theta^2} \right]$$

$$= (2\pi)^{-n/2} (a\theta^2)^{-n/2} \exp\left[- \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2 + n(\bar{x}_n - \theta)^2}{2a\theta^2} \right]$$

$$= \underbrace{(2\pi)^{-n/2} (a\theta^2)^{-n/2} \exp\left[- \frac{(n-1) S_n^2 + n(\bar{x}_n - \theta)^2}{2a\theta^2} \right]}_{g(S_n^2, \bar{x}_n; \theta)} \cdot \underbrace{1}_{h(\underset{\sim}{x})},$$

so $T(\underset{\sim}{X}) = (S_n^2, \bar{X}_n)$ is a suff. stat. for θ.

However, for any θ,

$$\mathbb{E}\, \bar{X}_n^2 = \text{Var}\, \bar{X}_n + (\mathbb{E}\bar{X}_n)^2 = a\theta^2 + \theta^2 = \theta^2(a+1)$$

and $\mathbb{E} S_n^2 = a\theta^2$,

so $\mathbb{E}\left[\left(\frac{a+1}{a}\right) S_n^2 - \bar{X}_n^2 \right] = 0$, but $P\left(\left(\frac{a+1}{a}\right) S_n^2 - \bar{X}_n^2 = 0 \right) \neq 1$.

Since we can find a function $g(T(\underset{\sim}{X}))$ such that $\mathbb{E}\, g(T(\underset{\sim}{X})) = 0$ for all $\theta \in \mathbb{R}$, $T(\underset{\sim}{X}) = (S_n^2, \bar{X}_n)$ is <u>not</u> a complete statistic.

$\boxed{6.20}$ Find a complete suff stat or show that it does not exist:

(a) Let $X_1, \ldots, X_n \overset{ind}{\sim} f(x;\theta) = \frac{2x}{\theta^2} \mathbb{1}(0 < x < \theta)$, $\theta > 0$.

We have

$$f(\underset{\sim}{x};\theta) = \underbrace{\left(\frac{2}{\theta^2}\right)^n \mathbb{1}(x_{(n)} < \theta)}_{g(x_{(n)};\theta)} \underbrace{\prod_{i=1}^n x_i \cdot \mathbb{1}(x_{(1)} > 0)}_{h(\underset{\sim}{x})},$$

so $T(\underset{\sim}{X}) = X_{(n)}$ is suff by the factorization theorem.

Now let $g(X_{(n)})$ satisfy $\mathbb{E}_\theta\, g(X_{(n)}) = 0 \quad \forall\, \theta$.

We have
$$X_{(n)} \sim f_{X_{(n)}}(x) = n\left[F_X(x;\theta)\right]^{n-1} f_X(x;\theta)$$
$$= n\left[\frac{x^2}{\theta^2}\right]^{n-1} \frac{2x}{\theta^2} \mathbb{1}(0 < x < \theta)$$
$$= \frac{2n}{\theta^{2n}} x^{2n-1} \mathbb{1}(0 < x < \theta),$$

so
$$0 = \mathbb{E}_\theta\, g(X_{(n)}) = \int_0^\theta g(x) \frac{2n}{\theta^{2n}} x^{2n-1}\, dx \quad \forall\, \theta$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial\theta} \mathbb{E}_\theta\, g(X_{(n)}) = \frac{\partial}{\partial\theta}\left[\frac{2n}{\theta^{2n}} \int_0^\theta g(x)\, x^{2n-1}\, dx\right]$$
$$= -\frac{1}{\theta^{2n+1}} \int_0^\theta g(x)\, x^{2n-1}\, dx + \frac{2n}{\theta^{2n}} g(\theta)\, \theta^{2n-1}$$
$$= -\frac{1}{2n\theta} \underbrace{\int_0^\theta \frac{2n}{\theta^{2n}} g(x)\, x^{2n-1}\, dx}_{=\mathbb{E}_\theta g(X_{(n)}) = 0} + \frac{2n}{\theta} g(\theta)$$
$$= \frac{2n}{\theta} g(\theta) \quad \forall\, \theta \quad \Rightarrow \quad g(\theta) = 0.$$

Therefore

$$P_\theta\left(g(X_{(n)}) = 0\right) = 1,$$

which means $T(\underset{\sim}{X}) = X_{(n)}$ is a complete sufficient statistic.

(b) Let $X_1, \dots, X_n \overset{ind}{\sim} f(x;\theta) = \dfrac{\theta}{(1+x)^{1+\theta}} \mathbb{1}(x>0)$, $\theta > 0$.

We can write

$$f(x;\theta) = \underbrace{\mathbb{1}(x>0)\left(\frac{1}{1+x}\right)}_{h(x)} \underbrace{\theta}_{c(\theta)} \exp\Big[\underbrace{-\theta}_{w_1(\theta)} \underbrace{\log(1+x)}_{t_1(x)}\Big].$$

number of functions

dimension of θ

so $f(x;\theta)$, $\theta>0$ is an exponential family (with $k=d$). Therefore

$$T(\underset{\sim}{X}) = \sum_{i=1}^n \log(1+X_i)$$

is a complete sufficient statistic.

(c) Let $X_1, \dots, X_n \overset{ind}{\sim} f(x;\theta) = \dfrac{(\log\theta)\,\theta^x}{\theta-1} \mathbb{1}(0<x<1)$, $\theta > 1$.

We can write

$$f(x;\theta) = \underbrace{\mathbb{1}(0<x<1)}_{h(x)} \underbrace{\frac{1}{\theta-1} \log\theta}_{c(\theta)} \cdot \exp\Big[\underbrace{x}_{t_1(x)} \underbrace{\log\theta}_{w_1(\theta)}\Big],$$

so $T(\underset{\sim}{X}) = \sum_{i=1}^n X_i$ is a complete sufficient statistic.

(d) Let $X_1, \dots, X_n \overset{ind}{\sim} f(x;\theta) = e^{-(x-\theta)} e^{-e^{-(x-\theta)}}$, $\theta \in \mathbb{R}$.

We can write

$$f(x;\theta) = \underbrace{e^{-x}}_{h(x)} \underbrace{e^{\theta}}_{c(\theta)} \cdot \exp\Big[\underbrace{-e^{\theta}}_{w_1(\theta) = -e^{\theta}} \underbrace{e^{x}}_{t_1(x) = e^x}\Big].$$

so $T(\underset{\sim}{X}) = \sum_{i=1}^n e^{X_i}$ is a complete sufficient statistic.

(e) Let $X_1, \ldots, X_n \sim f(x;\theta) = \binom{2}{x}\theta^x(1-\theta)^{2-x}\,\mathbb{1}(x \in \{0,1,2\})$.

We may write

$$f(x;\theta) = \underbrace{\mathbb{1}(x \in \{0,1,2\})\binom{2}{x}}_{h(x)}\,\underbrace{(1-\theta)}_{c(\theta)}\exp\Big[\underbrace{x}_{t_1(x)}\,\underbrace{\log\Big(\frac{\theta}{1-\theta}\Big)}_{w_1(\theta)}\Big],$$

from which we see that $T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is a complete suff. stat.

$\boxed{6.21}$ Let $X \sim f(x;\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}\,\mathbb{1}(x \in \{-1,0,1\})$, $0 \le \theta \le 1$.

(a) Is X a complete sufficient statistic for θ?

We know $T(X) = X$ is suff because there is no reduction of the data.

To check completeness, let $g(\cdot)$ be a function such that

$$\begin{aligned} \mathbb{E}\, g(X) &= g(-1)\left(\frac{\theta}{2}\right)^{|-1|}(1-\theta)^{1-|-1|} \\ &\quad + g(0)\left(\frac{\theta}{2}\right)^{|0|}(1-\theta)^{1-|0|} \\ &\quad + g(1)\left(\frac{\theta}{2}\right)^{|1|}(1-\theta)^{1-|1|} \\ &= [g(-1) + g(1)]\left(\frac{\theta}{2}\right) + g(0)(1-\theta) \\ &= 0 \quad \forall\, \theta. \end{aligned}$$

Suppose $g(x) = x$. Then $\mathbb{E}_\theta\, g(X) = 0 \ \forall\, \theta$ but $P_\theta(g(X)=0) = 1-\theta \ne 1$.

Therefore $T(X) = X$ is <u>not</u> a complete statistic.

(b) Is $T(X) = |X|$ a complete sufficient statistic for θ?

By the factorization result, we see that $T(X) = |X|$ is suff for θ.

To check completeness, let $g(\cdot)$ be a function such that

$$\mathbb{E}\, g(|X|) = g(|-1|)\left(\frac{\theta}{2}\right)^{|-1|}(1-\theta)^{1-|-1|}$$

$$+ g(0)\left(\frac{\theta}{2}\right)^{|0|}(1-\theta)^{1-|0|}$$

$$+ g(1)\left(\frac{\theta}{2}\right)^{|1|}(1-\theta)^{1-|1|}$$

$$= [g(1) + g(1)]\left(\frac{\theta}{2}\right) + g(0)(1-\theta)$$

$$= \theta\, g(1) + (1-\theta)\, g(0)$$

$$= g(0) + \theta\left(g(1) - g(0)\right)$$

$$= 0 \quad \forall\, \theta.$$

Setting $\theta = 1$ gives $g(1) = 0$ and setting $\theta = 0$ gives $g(0) = 0$.
So we have $P_\theta\left(g(|X|) = 0\right) = 1$.

Therefore $T(X) = |X|$ is a complete suff. statistic for θ.

(c) To see that $f(x;\theta)$ belongs to an exponential family, write

$$f(x;\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}\, \mathbb{1}(x \in \{-1,0,1\})$$

$$= \underbrace{\mathbb{1}(x \in \{-1,0,1\})\, 2^{-|x|}}_{h(x)}\, \underbrace{(1-\theta)}_{c(\theta)}\, \exp\Big[\underbrace{|x|}_{t(x)}\, \underbrace{\log\left(\frac{\theta}{1-\theta}\right)}_{w(\theta)}\Big].$$

$\boxed{6.22}$ Let $X_1, \ldots, X_n \overset{ind}{\sim} f(x;\theta) = \theta x^{\theta-1}\, \mathbb{1}(0<x<1)$, $\theta > 0$.

(a) The statistic $\sum_{i=1}^n X_i$ is <u>not</u> sufficient for θ.

We cannot factorize $f(\underset{\sim}{x};\theta)$ as $g\left(\sum_{i=1}^n x_i;\theta\right) h(\underset{\sim}{x})$.

(b) Write $f(x;\theta) = \underbrace{\mathbb{1}(0<x<1)\,\frac{1}{x}}_{h(x)} \cdot \underbrace{\theta}_{c(\theta)} \cdot \exp\Big[\underbrace{\theta}_{w_1(\theta)} \underbrace{\log x}_{t_1(x)}\Big].$

From here we see that $T(\underset{\sim}{X}) = \sum_{i=1}^n \log X_i$ is a comp. suff. stat.