STAT 713 hw 4

Rao-Blackwell, Fisher information, Cramér-Rao lower bound

Do problems 7.40, 7.41 from CB. In addition:

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\theta)$, $\theta > 0$, and consider estimating $\tau(\theta) = \theta(1+\theta)$.
   (a) Verify that $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ.
   (b) Propose an unbiased estimator $\tilde{\tau}$ of $\tau(\theta)$ based on X_1.
   (c) Now construct another unbiased estimator $\hat{\tau}$ for $\tau(\theta)$ by Rao-Blackwellization, that is, as $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$. *Hint: You will need to find the conditional pmf of X_1 given T.*
   (d) Is the estimator from part (c) the UMVUE for $\tau(\theta)$?

2. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Exponential}(\theta)$, $\theta > 0$, and consider estimating $\tau(\theta) = P_\theta(Y_1 \leq a) = 1 - e^{-a/\theta}$ for some $a > 0$.
   (a) Propose an unbiased estimator $\tilde{\tau}$ for τ.
   (b) Identify a complete sufficient statistic T for θ.
   (c) Obtain the UMVUE for τ by finding $\hat{\tau} = \mathbb{E}[\tilde{\tau}|T]$.

3. Let $Y_1, \dots, Y_n$ be independent rvs and $x_1, \dots, x_n$ known constants such that $Y_i = \beta x_i + \varepsilon_i$, for $i = 1, \dots, n$, where $\varepsilon_1, \dots, \varepsilon_n \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$, for some $\beta \in \mathbb{R}$ and $\sigma > 0$.
   (a) Find the Fisher information $I(\beta, \sigma^2)$. *You may want to put $\gamma = \sigma^2$ until your calculations are done.*
   (b) Give the CRLB for unbiased estimators of β.
   (c) Give the CRLB for unbiased estimators of σ^2.
   (d) Give the MLE for β and check whether it is unbiased.
   (e) Check whether the MLE for β achieves the CRLB.

4. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$.
   (a) Find the Fisher information $I(\alpha, \beta)$.
   Hint: Use $I(\theta) = -\mathbb{E}[\frac{\partial^2}{\partial \theta^2} \log(f(\mathbf{X}; \theta))]$ and let $\psi'(z) = \frac{d^2}{dz^2} \log \Gamma(z)$.
   (b) Find the CRLB for unbiased estimators of $\tau(\alpha, \beta) = \alpha\beta$.
   (c) Check whether $\hat{\tau} = \bar{X}_n$ achieves the CRLB for estimating $\tau(\alpha, \beta) = \alpha\beta$.

5. (Optional) For $t = 1, \dots, T$, let $X_t \sim \text{Normal}(\cos(2\pi t/T + \phi), 1)$, with $X_1, \dots, X_T$ indep., for some $\phi \in [0, 2\pi)$.
   (a) Show that the MLE for ϕ is the minimizer of the least-squares criterion

$$Q(\phi) = \sum_{t=1}^{T} \left(X_t - \cos(2\pi t/T + \phi)\right)^2.$$

(b) Show that the Fisher information is $I(\phi) = \sum_{t=1}^{T} \sin^2(2\pi t/T + \phi) = T/2$.
Hint: Use $\sin(2x) = 2\cos x \sin x$ *and* $\cos 2x = \cos^2 x - \sin^2 x$ *and use* $I(\phi) = -\mathbb{E}\frac{\partial^2}{\partial\phi^2} \log f(\mathbf{X}; \phi)$.

(c) For $\phi = 0$ and $\phi = \pi$, generate 500 data sets with $T = 10$ and compute the MLE for ϕ on each of the 500 simulated data sets. Make histograms of the 500 values of the MLE and report the variance of the MLE values. Does the MLE appear to achieve the CRLB for unbiased estimators of ϕ? Are the cases of $\phi = 0$ and $\phi = \pi$ different?