STAT 713 hw 5

Consistency, asymptotic distributions of estimators, asymptotic relative efficiency

Do problems 10.1, 10.10, 10.23 from CB. In addition:

1. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} F(x;\theta) = [1 + e^{-x/\theta}]^{-1}$ for some $\theta > 0$.
   (a) Give the asymptotic behavior of $\sqrt{n}(\hat{\theta}_n - \theta)$, where $\hat{\theta}_n$ is the MLE of θ.
   (b) Propose a variance stabilizing transformation of $\hat{\theta}_n$; that is, propose a function g such that $\sqrt{n}(g(\hat{\theta}_n) - g(\theta))$ has an asympototic variance which does not depend on θ.

2. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$, with $p \in (0,1)$, and let $\hat{p}_n = n^{-1}\sum_{i=1}^n X_i$.
   (a) Give the asymptotic behavior of $\sqrt{n}(\hat{p}_n - p)$ as $n \to \infty$.
   (b) Let $\tau = \tau(p) = \log(p/(1-p))$ be the log-odds and let $\hat{\tau}_n$ be the MLE of τ. Give the asymptotic behavior of $\sqrt{n}(\hat{\tau}_n - \tau)$ as $n \to \infty$.

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Beta}(\alpha, 1)$, for some $\alpha > 0$.
   (a) Let $\bar{\alpha}_n$ be the method of moments estimator for α. Find the asymptotic variance ϑ_1 such that $\sqrt{n}(\bar{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta_1)$ as $n \to \infty$.
   (b) Let $\hat{\alpha}_n$ be the maximum likelihood estimator for α. Find the asymptotic variance ϑ_2 such that $\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta_2)$ as $n \to \infty$.
   (c) Give the asymptotic relative efficiency $\text{ARE}(\bar{\alpha}_n; \hat{\alpha}_n)$ of the MoM estimator compared to the MLE.