STAT 713 hw 5

Consistency, asymptotic distributions of estimators, asymptotic relative efficiency

Do problems 10.1, 10.10, 10.23 from CB. In addition:

1. Let $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} F(x;\theta) = [1 + e^{-x/\theta}]^{-1}$ for some $\theta > 0$.

   (a) Give the asymptotic behavior of $\sqrt{n}(\hat{\theta}_n - \theta)$, where $\hat{\theta}_n$ is the MLE of θ. *Hint: You may need to use a computer to compute a complicated integral.*

The pdf is given by

$$f(x;\theta) = \frac{1}{\theta}\frac{e^{-x/\theta}}{[1 + e^{-x/\theta}]^2},$$

from which we in a few steps obtain

$$\frac{\partial}{\partial\theta}\log f(x;\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}\left(\frac{1 - e^{-x/\theta}}{1 + e^{-x/\theta}}\right).$$

From the above we note that

$$\mathbb{E}\left[\frac{X_1}{\theta}\left(\frac{1 - e^{-X_1/\theta}}{1 + e^{-X_1/\theta}}\right)\right] = 1,$$

since the score function has expected value equal to zero. To obtain the Fisher information for a sample with a single observation we write

$$\begin{aligned}
I_1(\theta) &= \operatorname{Var} S(\theta; X_1)\\
&= \frac{1}{\theta^2}\operatorname{Var}\left[Y\left(\frac{1 - e^{-Y}}{1 + e^{-Y}}\right)\right], \quad \text{where } Y = X_1/\theta \sim f_Y(y) = \frac{e^{-y}}{1 + e^{-y}}\\
&= \frac{1}{\theta^2}\left[\mathbb{E}Y^2\left(\frac{1 - e^{-Y}}{1 + e^{-Y}}\right)^2 - 1\right]\\
&= \frac{1}{\theta^2}\left[\int_{-\infty}^{\infty} y^2\left(\frac{1 - e^{-y}}{1 + e^{-y}}\right)^2\frac{e^{-y}}{(1 + e^{-y})^2}dy - 1\right]\\
&= \frac{1}{\theta^2}\int_{-1}^{1} 2\tanh^{-1}(u)u^2 du, \quad \text{with } u = \frac{1 - e^{-y}}{1 + e^{-y}} = \tanh(y/2)\\
&= \frac{1}{\theta^2}\left[\frac{1}{9}(12 + \pi^2) - 1\right]\\
&= \frac{1}{\theta^2}\left[\frac{3 + \pi^2}{9}\right],
\end{aligned}$$

where the value of the complicated integral can be obtained by typing

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integrate y**2*((1-exp(-y))/(1+exp(-y)))**2*exp(-y)/(1+exp(-y))**2 from -inf to inf
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or

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integrate 2*arctanh(u)**2 *u**2 from -1 to 1
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into WolframAlpha. Now we have

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{D}} \text{Normal}(0, 9\theta^2/(3+\pi^2))$$

as $n \to \infty$.

(b) Propose a variance stabilizing transformation of $\hat{\theta}_n$; that is, propose a function g such that $\sqrt{n}(g(\hat{\theta}_n) - g(\theta))$ has an asympototic variance which does not depend on θ.

We see that if we set $g(\theta) = \log\theta$, the delta method will give

$$\sqrt{n}(\log\hat{\theta}_n - \log\theta) \xrightarrow{\text{D}} \text{Normal}(0, 9/(3+\pi^2))$$

as $n \to \infty$, so the asymptotic variance does not depend on θ.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Bernoulli}(p)$, with $p \in (0,1)$, and let $\hat{p}_n = n^{-1}\sum_{i=1}^n X_i$.

(a) Give the asymptotic behavior of $\sqrt{n}(\hat{p}_n - p)$ as $n \to \infty$.

Since $p_n = \bar{X}_n$, $E[\bar{X}_n] = p$, and $Var(\bar{X}_n) = p(1-p)/n$, then by Central Limit Theorem, we have

$$\sqrt{n}(\hat{p}_n - p) \to N(0, p(1-p))$$

(b) Let $\tau = \tau(p) = \log(p/(1-p))$ be the log-odds and let $\hat{\tau}_n$ be the MLE of τ. Give the asymptotic behavior of $\sqrt{n}(\hat{\tau}_n - \tau)$ as $n \to \infty$.

Note that $\tau(p_n)$ is the MLE of $\tau(p)$. Thus, $\hat{\tau}_n = \log(\frac{\bar{X}_n}{1-\bar{X}_n})$
By delta method we can have:

$$\sqrt{n}(\tau(p_n) - \tau(p)) \to N(0, p(1-p)\tau'(p)^2)$$

where $\tau'(p) = \frac{1}{p(1-p)}$ Therefore:

$$\sqrt{n}(\hat{\tau} - \tau) \to N(0, 1/(p(1-p)))$$

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Beta}(\alpha, 1)$, for some $\alpha > 0$.

(a) Let $\bar{\alpha}_n$ be the method of moments estimator for α. Find the asymptotic variance ϑ_1 such that $\sqrt{n}(\bar{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta_1)$ as $n \to \infty$.

We have

$$m_1 = \frac{\alpha}{\alpha+1} \iff \alpha = \frac{m_1}{1-m_1}.$$

We begin with the fact that

$$\sqrt{n}(\hat{m}_1 - m_1) \xrightarrow{\text{D}} \text{Normal}(0, m_2 - m_1^2)$$

as $n \to \infty$, where $m_2 - m_1^2 = \frac{\alpha}{(\alpha+1)^2(\alpha+2)}$. Now, for $g(m_1) = m_1/(1-m_1) = \alpha$ we have $g'(m_1) = 1/(1-m_1)^2 = (\alpha+1)^2$. By the delta method, the asymptotic variance of $\sqrt{n}(\bar{\alpha}_n - \alpha)$ is given by

$$\vartheta_1 = [(\alpha+1)^2]^2 \frac{\alpha}{(\alpha+1)^2(\alpha+2)} = \frac{\alpha(\alpha+1)^2}{\alpha+2},$$

so we have

$$\sqrt{n}(\bar{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}\left(0, \frac{\alpha(\alpha+1)^2}{\alpha+2}\right)$$

as $n \to \infty$.

(b) Let $\hat{\alpha}_n$ be the maximum likelihood estimator for α. Find the asymptotic variance ϑ_2 such that $\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta_2)$ as $n \to \infty$.

The score function is given by

$$S(\alpha; \mathbf{X}) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log X_i,$$

giving the Fisher information

$$I(\alpha) = \text{Var}\, S(\alpha; \mathbf{X}) = n\, \text{Var}(\log X_1).$$

It can be shown that $Y = -\log X_1 \sim \text{Exponential}(1/\alpha)$, so that $\text{Var}(\log X_1) = 1/\alpha^2$. Therefore $I_1(\alpha) = 1/\alpha^2$. This gives

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{\text{D}} \text{Normal}\left(0, \alpha^2\right),$$

as $n \to \infty$. So $\vartheta_2 = \alpha^2$.

(c) Give the asymptotic relative efficiency $\text{ARE}(\bar{\alpha}_n; \hat{\alpha}_n)$ of the MoM estimator compared to the MLE.

The asymptotic relative efficiency of the MoM estimator compared to the MLE is

$$\text{ARE}(\bar{\alpha}_n; \hat{\alpha}_n) = \frac{\alpha(\alpha+2)}{(\alpha+1)^2} < 1$$

for all $\alpha > 0$. This means the MoM is not as efficient as the MLE.

Problems 10.1 and 10.10 from CB.

$\boxed{10.1}$ Let $X_1, \dots, X_n \overset{ind}{\sim} f_X(x;\theta) = \frac{1}{2}(1+\theta x)\mathbb{1}(-1 < x < 1)$, $-1 < \theta < 1$.

Consider the method of moments estimator for θ: We have

$$m_1 = \int_{-1}^{1} \frac{x}{2}(1+\theta x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{\theta x^3}{3}\right]\Bigg|_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2} + \frac{\theta}{3}\left[(1)-(-1)\right]\right]$$

$$= \frac{\theta}{3}.$$

So the M.o.M. estimator is given by

$$\bar{\theta}_n = 3\hat{m}_1 = 3\bar{X}_n.$$

We have $\bar{X}_n \xrightarrow{p} \mathbb{E}X_1 = \frac{\theta}{3}$ as long as $\operatorname{Var} X_1 < \infty$ (WLLN).

Since X_1 has bounded support, $\operatorname{Var} X_1 < \infty$. Therefore

$$\bar{\theta}_n \xrightarrow{p} \theta,$$

i.e., $\bar{\theta}_n$ is a consistent estimator.

$\boxed{10.10}$ Let $X_1, \dots, X_n \overset{ind}{\sim}$ Bernoulli(p). The MLE is $\hat{p}_n = \bar{X}_n$.

(a) The Fisher information

$$L(p;\underset{\sim}{X}) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^{n\bar{X}_n}(1-p)^{n-n\bar{X}_n}$$

$$\ell(p;\underset{\sim}{X}) = n\bar{X}_n \log p + (n-n\bar{X}_n)\log(1-p)$$

$$\frac{\partial}{\partial p}\ell(p;\underset{\sim}{X}) = \frac{n\bar{X}_n}{p} - \frac{n-n\bar{X}_n}{1-p}$$

$$= n\bar{x}_n\left(\frac{1}{p} + \frac{1}{1-p}\right) - \frac{n}{1-p}$$

$$= n\bar{x}_n\left(\frac{1}{p(1-p)}\right) - \frac{n}{1-p} .$$

The Fisher information is

$$I(p) = \mathrm{Var}\left[\frac{\partial}{\partial p} \ell(p; \underset{\sim}{x})\right]$$

$$= n^2\left(\mathrm{Var}\,\bar{x}_n\right) \frac{1}{p^2(1-p)^2}$$

$$= \frac{n}{p(1-p)} .$$

So the CRLB for unbiased estimators of $\tau(p) = p(1-p)$ is

$$\frac{\left[\tau'(p)\right]^2}{I_n(p)} = \frac{(1-2p)^2\, p(1-p)}{n} .$$

Now we have

$$\mathrm{Leff}\ \hat{\tau}_n = \lim_{n\to\infty} \frac{(1-2p)^2\, p(1-p)/n}{\mathrm{Var}\ \hat{\tau}_n}$$

$$= \lim_{n\to\infty} \frac{(1-2p)^2\, p(1-p)/n}{\frac{1}{n}\mathrm{Var}\left(\sqrt{n}\,\hat{\tau}_n\right)}$$

$$= \frac{(1-2p)^2\, p(1-p)}{\lim_{n\to\infty} \mathrm{Var}\left(\sqrt{n}\,\hat{\tau}_n\right)} \qquad \sqrt{n}\left(\hat{\tau}_n - \tau\right) \xrightarrow{D} N\left(0, (1-2p)^2 p(1-p)\right)$$

$$= 1 ,$$

so $\hat{\tau}_n$ is asymptotically efficient provided $p \neq 1/2$.

(b) If $p = 1/2$, then

$$n\left(\hat{p}_n(1-\hat{p}_n) - p(1-p)\right) \xrightarrow{D} c''(p)\,\frac{p(1-p)}{2}\cdot W, \qquad W \sim \chi^2_1,$$

$$= -p(1-p)\cdot W,$$

since $c''(p) = -2$. Note that $\mathrm{Var}\left(-p(1-p)W\right) = \frac{1}{16}\,\mathrm{Var}\,W = \frac{2}{16} = \frac{1}{8}$.

(c) Suppose $p = 1/2$. Then

$$\mathrm{Var}\left(\hat{p}_n(1-\hat{p}_n)\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)$$

$$= \frac{(n-1)^2}{n^2}\,\mathrm{Var}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)$$

$$= \left(\frac{n-1}{n}\right)^2 \mathrm{Var}\; S_n^2$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{1}{n}\left[\theta_4 - \left(\frac{n-3}{n-1}\right)\theta_2^2\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{1}{n}\left[\frac{1}{16} - \left(\frac{n-3}{n-1}\right)\frac{1}{16}\right]$$

$$= \frac{(n-1)}{n^3}\left[\frac{(n-1)-(n-3)}{16}\right]$$

$$= \frac{n-1}{n^3}\,\frac{1}{8},$$

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$\theta_4 = \mathbb{E}(X_1 - p)^4$

$\theta_2^2 = [\mathbb{E}(X_1-p)^2]^2$

where

$$\theta_2^2 = [p(1-p)]^2 = \left[\tfrac{1}{2}\left(1-\tfrac{1}{2}\right)\right]^2 = \frac{1}{16}$$

$$\theta_4 = \mathbb{E}\left[X_1^4 - 4X_1^3 p + 6X_1^2p^2 - 4X_1p^3 + p^4\right]$$

$$= p - 4p^2 + 6p^3 - 4p^4 + p^4$$

$$= p - 4p^2 + 6p^3 - 3p^4$$

$$= \frac{1}{2} - 4\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{2} - 1 + \frac{6}{8} - \frac{3}{16}$$

$$= \frac{1}{16}.$$

Our work shows that when $p = 1/2$,

$$\mathrm{Var}\left(\sqrt{n}\ \hat{p}_n(1-\hat{p}_n)\right) = \frac{(n-1)}{n^2}\frac{1}{8} \to 0,$$

whereas $$\mathrm{Var}\left(n\ \hat{p}_n(1-\hat{p}_n)\right) = \left(\frac{n-1}{n}\right)\frac{1}{8} \to \frac{1}{8}.$$