STAT 713 hw 6

Asymptotic distributions of estimators, size and power of some classical tests

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda)$.
   (a) Find the ARE of $\bar{X}_n$ with respect to $\hat{\sigma}^2_n = n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ for estimating λ.
   (b) Which estimator is more "efficient"?

2. Let Z and W be independent rvs such that $Z(0,1)$ and $W \sim \chi^2_\nu$ and let ϕ be a constant. Then

$$\frac{Z + \phi}{\sqrt{W/\nu}} \sim t_{\nu,\phi},$$

   where $t_{\nu,\phi}$ denotes the non-central t-distribution with degrees of freedom ν and non-centrality parameter ϕ.

   Now, suppose $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$, and let μ_0 be a fixed constant.
   (a) Give the distribution of $\sqrt{n}(\bar{X}_n - \mu_0)/S_n$.
   (b) Letting $F_{\nu,\phi}$ denote the cdf of the $t_{\nu,\phi}$ distribution, give the power function for the test of H_0: $\mu = \mu_0$ vs H_1: $\mu \neq \mu_0$ with decision rule $\phi(\mathbf{X}) = 1 \iff \sqrt{n}|\bar{X}_n - \mu_0|/S_n > t_{n-1,\alpha/2}$.
   (c) Suppose $\sigma = 1$, $\mu_0 = 3$, and $\alpha = 0.05$.
      i. Use R to make a plot of the power function in part (b) under $n = 5$.
      ii. Suppose $\mu = 4$. What is the smallest sample size n under which the test will reject H_0 with probability at least 0.90?

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and consider the test of H_0: $\sigma^2 \leq \sigma_0^2$ versus H_1: $\sigma^2 > \sigma_0^2$ with decision rule $\phi(\mathbf{X}) = 1 \iff (n-1)S_n^2/\sigma_0^2 > c$.
   (a) Give an expression for the power function $\gamma(\sigma^2)$ of the test.
   (b) Find c such that the test has size α.
   (c) Suppose $\sigma_0^2 = 2$ and $\alpha = 0.05$.
      i. Use R to make a plot of the power curve of the test under $n = 10$.
      ii. Suppose $\sigma^2 = 2.1$. Find the sample size necessary to reject the null hypothesis with probability at least 0.80.

4. Let $X_{i1}, \dots, X_{in_i} \overset{\text{ind}}{\sim} \text{Normal}(\mu_i, \sigma^2)$, $i = 1, 2$, be two independent random samples. Let $\bar{X}_1$, $\bar{X}_2$, S_1^2, and S_2^2 be the means and variances of the two samples, respectively. Moreover, let $S^2_{\text{pooled}} = ((n_1 - 1)S_1^2 + (n_2 - 1)S_2^2)/(n_1 + n_2 - 2)$.
   (a) Give the distribution of

$$\frac{\bar{X}_1 - \bar{X}_2 - \delta}{S_{\text{pooled}}\sqrt{1/n_1 + 1/n_2}},$$

   where δ is a constant.

(b) For testing H_0: $\mu_1 - \mu_2 = 0$ versus H_1: $\mu_1 - \mu_2 \neq 0$, give the value of c such that the test with decision rule

$$\phi(\mathbf{X}) = 1 \iff \frac{|\bar{X}_1 - \bar{X}_2|}{S_{\text{pooled}}\sqrt{1/n_1 + 1/n_2}} > c$$

has size α.

(c) Use R to plot the power curve as of the test in (b) with size $\alpha = 0.05$ under $n_1 = 10$, $n_2 = 15$, and $\sigma^2 = 4$. Make the power a function of $\delta = \mu_1 - \mu_2$.

5. Let $X_{i1}, \dots, X_{in_i} \overset{\text{ind}}{\sim} \text{Normal}(\mu_i, \sigma_i^2)$, $i = 1, 2$, be two independent random samples. Let $\bar{X}_1$, $\bar{X}_2$, S_1^2, and S_2^2 be the means and variances of the two samples, respectively.

(a) Give the distribution of $(S_1^2/\sigma_1^2)/(S_2^2/\sigma_2^2)$.

(b) Consider testing the hypotheses H_0: $\sigma_1^2/\sigma_2^2 \geq \vartheta_0$ versus H_1: $\sigma_1^2/\sigma_2^2 < \vartheta_0$ for some constant ϑ_0. Give an expression for the power function of the test with decision rule $\phi(\mathbf{X}) = 1 \iff (S_1^2/S_2^2)/\vartheta_0 < c$. Express the power as a function of $\vartheta = \sigma_1^2/\sigma_2^2$.

(c) Find c such that the test in (b) has size α.

(d) Under $\alpha = 0.05$, $n_1 = 15$, and $n_2 = 5$, use R to plot the power function over a range of ϑ values when testing for equal variances.