STAT 713 hw 7

Likelihood ratio tests

Do problems 8.3, 8.5, 8.6, 8.31, 8.41 from CB. In addition:

1. Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$, where α and β are unknown.
   (a) Give the likelihood function $L(\alpha, \beta; \mathbf{X})$, where $\mathbf{X} = (X_1, \dots, X_n)$.
   (b) Give the log-likelihood function $\ell(\alpha, \beta; \mathbf{X})$.
   (c) For any $\alpha > 0$, give an expression for $\hat{\beta}(\alpha) = \text{argmax}_\beta \, \mathcal{L}(\alpha, \beta; \mathbf{X})$.
   (d) Consider testing the hypotheses H_0: $\alpha = \alpha_0$ versus H_1: $\alpha \neq \alpha_0$ and let $\hat{\alpha}$ be the maximum likelihood estimator for α. Given an expression for $-2 \log \text{LR}(\mathbf{X})$, where $\text{LR}(\mathbf{X})$ is the likelihood ratio.
   (e) The following R code stores in the vector `X` the survival times of several guinea pigs from the point in time at which they were infected with virulent tubercle bacilli and computes on these data the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ for the $\text{Gammma}(\alpha, \beta)$ distribution. The data are taken from Bjerkedal (1960).

```
X <-   c(12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52,
        53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62,
        63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84,
        85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131,
        143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376)

library(MASS) # pull in library of functions including the fitdistr() function
fitdistr(X,"gamma") # gives alpha.hat and 1/beta.hat
```

   Compute $-2 \log \text{LR}(\mathbf{X})$ for these data when testing H_0: $\alpha = 1$ versus $H_0 : \alpha \neq 1$.
   (f) Report the p-value of the asymptotic likelihood ratio test of H_0: $\alpha = 1$ versus $H_0 : \alpha \neq 1$.
   (g) Consider testing H_0: $\alpha = \alpha_0$ versus H_1: $\alpha \neq \alpha_0$ using the guinea pig data. Find an interval such that you fail to reject H_0 at the 0.01 significance level for all α_0 in the interval. *Hint: Compute $-2 \log \text{LR}(\mathbf{X})$ over many values of α_0 and find those values of α_0 (search, say, between $1/2$ and 4) for which $-2 \log \text{LR}(\mathbf{X}) < \chi^2_{1,0.01}$.*
   (h) Give an interpretation of this interval.
   (i) Based on these results, do you think it would be reasonable to model these data using the $\text{Exponential}(\beta)$ distribution?

References

Bjerkedal, T. (1960). Acquisition of Resistance in Guinea Pigs infected with Different Doses of Virulent Tubercle Bacilli. *American Journal of Hygiene*, 72(1), 130-48.