STAT 713 hw 7

Likelihood ratio tests

Do problems 8.3, 8.5, 8.6, 8.31, 8.41 from CB. In addition:

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha, \beta)$, where α and β are unknown.

   (a) Give the likelihood function $L(\alpha, \beta; \mathbf{X})$, where $\mathbf{X} = (X_1, \dots, X_n)$.

   The likelihood function is given by

$$
\begin{aligned}
L(\alpha, \beta; \mathbf{X}) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} X_i^{\alpha-1} \exp(-X_i/\beta) \\
&= \Gamma(\alpha)^{-n} \beta^{-n\alpha} \left(\prod_{i=1}^{n} X_i \right)^{\alpha-1} \exp\left(-\sum_{i=1}^{n} X_i/\beta \right)
\end{aligned}
$$

   (b) Give the log-likelihood function $\ell(\alpha, \beta; \mathbf{X})$.

   The log-likelihood function is given by

$$
\ell(\alpha, \beta; X_1, \dots, X_n) = -n \log \Gamma(\alpha) - n\alpha \log \beta + (\alpha - 1) \sum_{i=1}^{n} \log X_i - \sum_{i=1}^{n} X_i/\beta
$$

   (c) For any $\alpha > 0$, give an expression for $\hat{\beta}(\alpha) = \text{argmax}_{\beta}\, \mathcal{L}(\alpha, \beta; \mathbf{X})$.

   We get $\hat{\beta}(\alpha) = \bar{X}_n/\alpha$.

   (d) Consider testing the hypotheses H_0: $\alpha = \alpha_0$ versus H_1: $\alpha \neq \alpha_0$ and let $\hat{\alpha}$ be the maximum likelihood estimator for α. Given an expression for $-2 \log \text{LR}(\mathbf{X})$, where $\text{LR}(\mathbf{X})$ is the likelihood ratio.

   The likelihood ratio is given by

$$
\begin{aligned}
\text{LR}(\mathbf{X}) &= \frac{\sup_{\{\alpha,\beta:\alpha=\alpha_0,\beta\geq 0\}} L(\alpha, \beta; \mathbf{X})}{\sup_{\{\alpha,\beta:\alpha\geq 0,\beta\geq 0\}} L(\alpha, \beta; \mathbf{X})} \\
&= \frac{L(\alpha_0, \hat{\beta}(\alpha_0); \mathbf{X})}{L(\hat{\alpha}, \hat{\beta}(\hat{\alpha}); \mathbf{X})}.
\end{aligned}
$$

From here we obtain

$$
\begin{aligned}
&-2\log \mathrm{LR}(\mathbf{X}) \\
&\quad = -2\left[n\log\left(\frac{\Gamma(\hat{\alpha})}{\Gamma(\alpha_0)}\right) + n(\hat{\alpha}-\alpha_0)\left(\log \bar{X}_n - n^{-1}\sum_{i=1}^{n}\log X_i + 1\right) + n\alpha_0\log\alpha_0 - n\hat{\alpha}\log\hat{\alpha}\right].
\end{aligned}
$$

(e) The following R code stores in the vector `X` the survival times of several guinea pigs from the point in time at which they were infected with virulent tubercle bacilli and computes on these data the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ for the $\text{Gammma}(\alpha, \beta)$ distribution. The data are taken from Bjerkedal (1960).

```
X <-   c(12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52,
        53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62,
        63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84,
        85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131,
        143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376)

library(MASS) # pull in library of functions including the fitdistr() function
fitdistr(X,"gamma") # gives alpha.hat and 1/beta.hat
```

Compute $-2\log \mathrm{LR}(\mathbf{X})$ for these data when testing H_0: $\alpha = 1$ versus $H_0 : \alpha \neq 1$.

From the R code

```
a.hat <- fitdistr(X,"gamma")$estimate[1]
a.0 <- 1
n <- length(X)

minus2ll <- -2*(n*log(gamma(a.hat)/gamma(a.0))+n*(a.hat-a.0)*
             (log(mean(X))-mean(log(X))+1)+n*a.0*log(a.0)-n*a.hat*log(a.hat))
```

we get the value 18.38911.

(f) Report the p-value of the asymptotic likelihood ratio test of H_0: $\alpha = 1$ versus $H_0 : \alpha \neq 1$.

We compute the area under the χ^2_1 distribution to the right of the value 18.38911. It is `1-pchisq(minus2ll,1)` $= 1.800847 \times 10^{-5}$.

(g) Consider testing H_0: $\alpha = \alpha_0$ versus H_1: $\alpha \neq \alpha_0$ using the guinea pig data. Find an interval such that you fail to reject H_0 at the 0.01 significance level for all α_0 in the interval. *Hint: Compute $-2\log \mathrm{LR}(\mathbf{X})$ over many values of α_0 and find those values of α_0 (search, say, between $1/2$ and 4) for which $-2\log \mathrm{LR}(\mathbf{X}) < \chi^2_{1,0.01}$.*

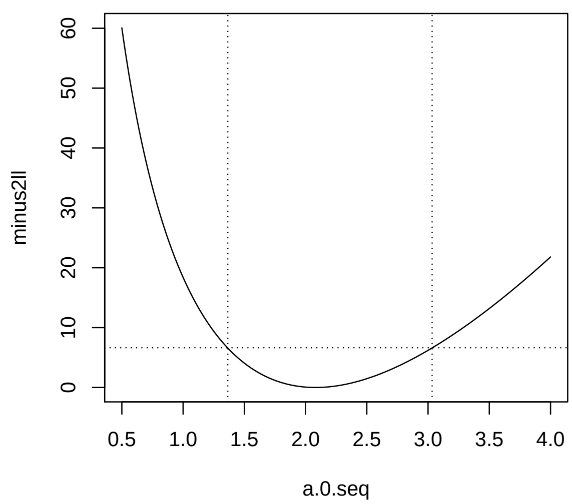
The following R code finds such an interval and also plots the $-2 \log \mathrm{LR}(\mathbf{X})$ across values of α_0 from $1/2$ to 4.

```
a.0.seq <- seq(1/2,4,length=1000)
minus2ll <- - 2 * (n*log( gamma(a.hat)/gamma(a.0.seq)) + n*(a.hat-a.0.seq)*
    (log(mean(X))-mean(log(X))+1)+n*a.0.seq*log(a.0.seq)-n*a.hat*log(a.hat))

plot(minus2ll~a.0.seq,type="l")
abline(h=qchisq(.99,1),lty=3)

which(minus2ll < qchisq(.99,1))
lower <- min(a.0.seq[which(minus2ll < qchisq(.99,1))])
upper <- max(a.0.seq[which(minus2ll < qchisq(.99,1))])

abline(v=lower,lty=3)
abline(v=upper,lty=3)
```



We have that $-2 \log \mathrm{LR}(\mathbf{X}) < \chi^2_{1,0.01}$ when $\alpha_0 \in (1.37, 3.03)$. Note that the interval is approximate.

(h) Give an interpretation of this interval.

This is a 99% confidence interval for α.

(i) Based on these results, do you think it would be reasonable to model these data using the Exponential(β) distribution?

The Exponential(β) distribution is the Gamma(α, β) distribution when $\alpha = 1$, and we reject H_0: $\alpha = 1$ at all significance levels greater than the p-value 1.800847×10^{-5}. Thus the evidence is quite strong that α is not equal to 1.

References

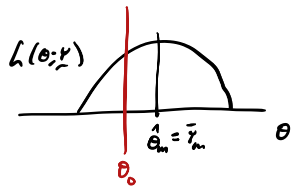
Bjerkedal, T. (1960). Acquisition of Resistance in Guinea Pigs infected with Different Doses of Virulent Tubercle Bacilli. *American Journal of Hygiene*, 72(1), 130-48.

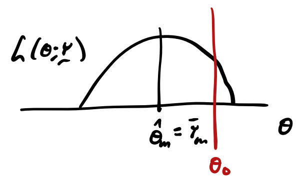
Problems 8.3, 8.5, 8.6, 8.31, and 8.41 from C.B.

$\boxed{8.3}$ Let $Y_1, \dots, Y_n \overset{ind}{\sim}$ Bernoulli(θ). Test $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$.

The likelihood function is $L(\theta; \underset{\sim}{Y}) = \prod_{i=1}^{n} \theta^{Y_i}(1-\theta)^{1-Y_i} = \theta^{n\bar{Y}_n}(1-\theta)^{n - n\bar{Y}_n}$.

Likelihood ratio is





$$LR(\underset{\sim}{Y}) = \frac{\sup_{\theta \le \theta_0} L(\theta; \underset{\sim}{Y})}{\sup_{\theta \in [0,1]} L(\theta; \underset{\sim}{Y})}$$

$$= \frac{L(\hat{\theta}_0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} \qquad \hat{\theta}_0 = \begin{cases} \hat{\theta}_n & \text{if } \hat{\theta}_n \le \theta_0 \\ \theta_0 & \text{if } \hat{\theta}_n > \theta_0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } \hat{\theta}_n \le \theta_0 \\ \dfrac{L(\theta_0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} & \text{if } \hat{\theta}_n > \theta_0 \end{cases}.$$

where

$$\frac{L(\theta_0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} = \frac{\theta_0^{n\bar{Y}_n}(1-\theta_0)^{n-n\bar{Y}_n}}{\bar{Y}_n^{n\bar{Y}_n}(1-\bar{Y}_n)^{n-n\bar{Y}_n}} = \left[\left(\frac{\theta_0}{\bar{Y}_n}\right)^{\bar{Y}_n}\left(\frac{1-\theta_0}{1-\bar{Y}_n}\right)^{1-\bar{Y}_n}\right]^n.$$

The LRT rejects H_0 when $LR(\underset{\sim}{Y}) < k$.

Note that

$$\log \frac{L(\theta_0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} = n\left\{\bar{Y}_n \log \theta_0 - \bar{Y}_n \log \bar{Y}_n + (1-\bar{Y}_n)\log(1-\theta_0) - (1-\bar{Y}_n)\log(1-\bar{Y}_n)\right\}$$

$$= n\left\{\bar{Y}_n \log\left(\frac{\theta_0}{1-\theta_0}\right) - \bar{Y}_n \log\left(\frac{\bar{Y}_n}{1-\bar{Y}_n}\right) + \log(1-\theta_0) - \log(1-\bar{Y}_n)\right\},$$

and

$$\frac{\partial}{\partial \bar{Y}_n} \log \frac{L(\theta_0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} = n\left\{\log\left(\frac{\theta_0}{1-\theta_0}\right) - \left[\log\left(\frac{\bar{Y}_n}{1-\bar{Y}_n}\right) + \bar{Y}_n\left(\frac{1-\bar{Y}_n}{\bar{Y}_n}\right)\left[\frac{1}{1-\bar{Y}_n} + \frac{\bar{Y}_n}{(1-\bar{Y}_n)^2}\right]\right] + \frac{1}{1-\bar{Y}_n}\right\}$$

$$\frac{d}{d\bar{y}_m}\frac{\bar{y}_m}{1-\bar{y}_m} = \frac{1}{1-\bar{y}_m} + \frac{\bar{y}_m}{(1-\bar{y}_m)^2}$$

$$= m\left\{\log\left(\frac{\theta_0}{1-\theta_0}\right) - \left[\log\left(\frac{\bar{y}_m}{1-\bar{y}_m}\right) + \frac{1}{1-\bar{y}_m}\right] + \frac{1}{1-\bar{y}_m}\right\}$$

$$= m\left\{\log\left(\frac{\theta_0}{1-\theta_0}\right) - \log\left(\frac{\bar{y}_m}{1-\bar{y}_m}\right)\right\}$$

$$= m\left\{\log\theta_0 - \log\bar{y}_m + \log(1-\bar{y}_m) - \log(1-\theta_0)\right\}$$

$$< 0 \quad \text{if} \quad \bar{y}_m > \theta_0.$$

This work shows us that $LR(\underset{\sim}{y})$ is decreasing in $\bar{y}_m$ for $\bar{y}_m > \theta_0$.

So there exists some k^* such that

$$LR(\underset{\sim}{y}) < k \iff \bar{y}_m > k^* \quad \left(\text{or } \sum_{i=1}^{m} y_i > k^*\right).$$

$\boxed{8.5}$ Let $X_1,\dots,X_n \overset{\text{ind}}{\sim} f_X(x;\theta,\nu) = \dfrac{\theta\nu^\theta}{x^{\theta+1}}\mathbb{1}(x\geq\nu)$, $\theta>0$, $\nu>0$.

(a) The likelihood function is given by

$$L(\theta,\nu;\underset{\sim}{x}) = \frac{\theta^n\nu^{n\theta}}{\left(\prod_{i=1}^n x_i\right)^{\theta+1}}\mathbb{1}(x_{(1)}\geq\nu)$$

For every $\theta>0$, $L(\theta,\nu;\underset{\sim}{x})$ is maximized at $\nu = x_{(1)}$, so

$$\hat{\nu}_n = x_{(1)}.$$

Then

$$\hat{\theta}_n = \underset{\theta>0}{\operatorname{argmax}}\; L(\theta, x_{(1)};\underset{\sim}{x}) = \underset{\theta>0}{\operatorname{argmax}}\; \frac{\theta^n x_{(1)}^{n\theta}}{\left(\prod_{i=1}^n x_i\right)^{\theta+1}}.$$

We have

$$\ell(\theta, x_{(1)}; \underset{\sim}{x}) = n \log \theta + n\theta \log x_{(1)} - (\theta+1) \sum_{i=1}^{n} \log x_i ,$$

and

$$\frac{\partial}{\partial\theta} \ell(\theta, x_{(1)}; \underset{\sim}{x}) = \frac{n}{\theta} + n \log x_{(1)} - \sum_{i=1}^{n} \log x_i$$

$$= \frac{n}{\theta} - \sum_{i=1}^{n} \left(\log x_i - \log x_{(1)} \right)$$

$$\overset{\text{set}}{=} 0 .$$

This gives

$$\hat{\theta}_n = \frac{n}{\sum_{i=1}^{n} \left[\log x_i - \log x_{(1)} \right]} = \frac{n}{\log \left(\prod_{i=1}^{n} \frac{x_i}{x_{(1)}} \right)}$$

(b) Test $H_0: \theta = 1$ vs $H_1: \theta \neq 1$.

The LR is

$$LR(\underset{\sim}{x}) = \frac{\sup_{\theta=1,\, \nu>0} L(\theta, \nu; \underset{\sim}{x})}{\sup_{\theta>0,\, \nu>0} L(\theta, \nu; \underset{\sim}{x})}$$

$$= \frac{L(1, x_{(1)}; \underset{\sim}{x})}{L(\hat{\theta}_n, x_{(1)}; \underset{\sim}{x})}$$

$$\frac{\theta^n \nu^{n\theta}}{\left(\prod_{i=1}^{n} x_i\right)^{\theta+1}}$$

$$= \frac{1^n x_{(1)}^{n \cdot 1}}{\left(\prod_{i=1}^{n} x_i\right)^{1+1}} \mathbb{1}(x_{(1)} \geq x_{(1)}) \Bigg/ \left[\frac{\hat{\theta}_n^n x_{(1)}^{n\hat{\theta}_n}}{\left(\prod_{i=1}^{n} x_i\right)^{\hat{\theta}_n+1}} \mathbb{1}(x_{(1)} \geq x_{(1)}) \right]$$

$$= \frac{1}{\hat{\theta}_n^n} x_{(1)}^{n - n\hat{\theta}_n} \left(\prod_{i=1}^{n} x_i \right)^{\hat{\theta}_n + 1 - (1+1)}$$

$$= \frac{1}{\hat{\theta}_n^n} \left[x_{(1)}^{1-\hat{\theta}_n} \right]^n \left(\prod_{i=1}^{n} x_i \right)^{-(1-\hat{\theta}_n)}$$

$$= \left(\frac{1}{\hat{\theta}_n}\right)^n \left[\prod_{i=1}^{n} \frac{X_i}{X_{(1)}}\right]^{(\hat{\theta}_n - 1)}$$

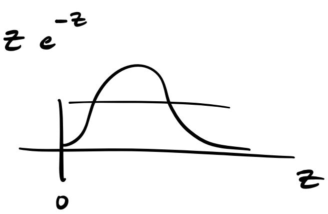
$$= \left(\frac{1}{\hat{\theta}_n}\right)^n \left[\exp\left(n \underbrace{\frac{1}{n} \log\left(\prod_{i=1}^{n} \frac{X_i}{X_{(1)}}\right)}_{1/\hat{\theta}_n}\right)\right]^{\hat{\theta}_n - 1}$$

$$= \left(\frac{1}{\hat{\theta}_n}\right)^n \exp\left[\frac{1}{\hat{\theta}_n} (\hat{\theta}_n - 1)\right]^n$$

$$= \left(\frac{1}{\hat{\theta}_n}\right)^n \exp\left(-\frac{1}{\hat{\theta}_n}\right)^n e^n$$

The LRT rejects when

$$\left(\frac{1}{\hat{\theta}_n}\right)^n \exp\left(-\frac{1}{\hat{\theta}_n}\right)^n e^n < k$$



$\iff \quad \frac{1}{\hat{\theta}_n} < k_1 \quad$ or $\quad \frac{1}{\hat{\theta}_n} > k_2 \quad$ since $z e^{-z}$ has this shape

$\iff \quad \frac{n}{\hat{\theta}_n} < n k_1 \quad$ or $\quad \frac{n}{\hat{\theta}_n} > n k_2$,

where

$$\frac{n}{\hat{\theta}_n} = \log\left(\prod_{i=1}^{n} \frac{X_i}{X_{(1)}}\right) .$$

(c) Under $H_0: \theta = 1$, $\quad X_1, \dots, X_n \overset{iid}{\sim} \frac{v}{x^2} \mathbb{1}(x \geq v)$.

Note that

$$\log\left(\prod_{i=1}^{n} \frac{X_i}{X_{(1)}}\right) = \log\left(\prod_{i=2}^{n} \frac{X_{(i)}}{X_{(1)}}\right) = \sum_{i=2}^{n} \log\left(\frac{X_{(i)}}{X_{(1)}}\right).$$

The joint density of $X_{(1)}, \dots, X_{(n)}$ is

$$f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n) = n! \cdot \prod_{i=1}^{n} \frac{\nu}{x_i^2} \mathbb{1}(x_i \geq \nu) \quad . \quad \left(\text{pg. } 230 \text{ of CB}\right)$$

Define $Y_1 = X_{(1)}$ and $Y_i = X_{(i)}/X_{(1)}$ for $i = 2, \dots, n$. Then we have

$$\begin{aligned} y_1 &= x_1 & & & x_1 &= y_1 \\ y_2 &= \frac{x_2}{x_1} & &\iff & x_2 &= y_1 y_2 \\ &\vdots & & & &\vdots \\ y_n &= \frac{x_n}{x_1} & & & x_n &= y_1 y_n \end{aligned}$$

$$J(y_1,\dots,y_n) = \begin{vmatrix} 1 & y_2 & y_3 & \cdots & y_n \\ 0 & y_1 & 0 & \cdots & 0 \\ 0 & 0 & y_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_1 \end{vmatrix} = y_1^{n-1} .$$

So the joint density of $(Y_1,\dots,Y_n)$ is given by

$$f_{Y_1,\dots,Y_n}(y_1,\dots,y_n) = n! \frac{\nu}{y_1} \prod_{i=2}^{n} \frac{\nu}{(y_1 y_i)^2} \, y_1^{n-1} \, \mathbb{1}(y_1 \geq \nu) \, \mathbb{1}(1 \leq y_2 < \dots < y_n)$$

$$= n! \frac{\nu^n}{y_1^n} \prod_{i=2}^{n} \frac{1}{y_i^2} \, \mathbb{1}(y_1 \geq \nu) \, \mathbb{1}(1 \leq y_2 < \dots < y_n)$$

$$= n \frac{\nu^n}{y_1^n} \mathbb{1}(y_1 \geq \nu) * (n-1)! \prod_{i=2}^{n} \frac{1}{y_i^2} \, \mathbb{1}(1 \leq y_2 < \dots < y_n)$$

from which we see that $Y_1 \perp\!\!\!\perp (Y_2,\dots,Y_n)$.

We may regard $Y_2,\dots,Y_n$ as the order statistics of a random sample from the dist. with pdf $f_Y(y) = \frac{1}{y^2} \mathbb{1}(y \geq 1)$.

We wish to find the distribution of $2 \cdot \log\left(\prod_{i=1}^{n} \frac{X_{(i)}}{X_{(1)}}\right) = 2 \sum_{i=1}^{n} \log Y_i$.

If $\quad Y \sim f_Y(y) = \frac{1}{y^2}\, \mathbb{1}(y > 1),\quad$ then

$$W = \log Y \sim f_W(w) = \frac{1}{(e^w)^2}\, e^w\, \mathbb{1}(w > 0) = e^{-w}\, \mathbb{1}(w > 0).$$

$$\left(w = \log y \iff y = e^w,\ \frac{dy}{dw} = e^w \right),$$

so that $\quad \log Y \sim \text{Exponential}(1).$

From here we obtain

$$\underbrace{2 \sum_{i=2}^{n} \log Y_i}\ \sim\ \text{Gamma}(n-1, 2) = \text{Gamma}\left(\frac{2(n-1)}{2}, 2\right) \sim \chi^2_{2(n-1)}$$

the sum of all the order statistics $Y_{(2)}, \dots, Y_{(n)}$ has the same dist as the sum of the unordered values.

$\boxed{8.6}$ $\quad X_1, \dots, X_n \overset{ind}{\sim} \text{Exponential}(\theta), \quad Y_1, \dots, Y_m \overset{ind}{\sim} \text{Exponential}(\mu).$

(a) Find LRT for $H_0: \theta = \mu$ vs $H_1: \theta \neq \mu$.

$$LR(\underset{\sim}{X}, \underset{\sim}{Y}) = \frac{\sup_{\theta = \mu} L(\theta, \mu; \underset{\sim}{X}, \underset{\sim}{Y})}{\sup_{\theta > 0, \mu > 0} L(\theta, \mu; \underset{\sim}{X}, \underset{\sim}{Y})}$$

$$= \frac{L(\hat{\theta}_0, \hat{\theta}_0; \underset{\sim}{X}, \underset{\sim}{Y})}{L(\hat{\theta}, \hat{\mu}; \underset{\sim}{X}, \underset{\sim}{Y})},$$

where $\hat{\theta} = \bar{X}_n$, $\hat{\mu} = \bar{Y}_m$, and

$$\hat{\theta}_0 = \underset{\theta > 0}{\text{argmax}}\ L(\theta, \theta; \underset{\sim}{X}, \underset{\sim}{Y})$$

$$= \underset{\theta > 0}{\text{argmax}} \left(\frac{1}{\theta}\right)^n e^{-\frac{n\bar{X}_n}{\theta}} \left(\frac{1}{\theta}\right)^m e^{-\frac{m\bar{Y}_m}{\theta}}$$

$$= \operatorname*{argmax}_{\theta > 0} \left(\frac{1}{\theta}\right)^{n+m} e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j\right)}$$

just combine the data

$$= \frac{1}{n+m}\left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j\right).$$

$$= \frac{n\bar{X}_n + m\bar{Y}_m}{n+m}.$$

So we have

$$LR(\underset{\sim}{X}, \underset{\sim}{Y}) = \frac{\left(\frac{n+m}{n\bar{X}_n + m\bar{Y}_m}\right)^{n} e^{-n\bar{X}_n\left(\frac{n+m}{n\bar{X}_n + m\bar{Y}_m}\right)} \left(\frac{n+m}{n\bar{X}_n + m\bar{Y}_m}\right)^{m} e^{-m\bar{Y}_m\left(\frac{n+m}{n\bar{X}_n + m\bar{Y}_m}\right)}}{\left(\frac{1}{\bar{X}_n}\right)^{n} e^{-\frac{n\bar{X}_n}{\bar{X}_n}} \left(\frac{1}{\bar{Y}_m}\right)^{m} e^{-\frac{m\bar{Y}_m}{\bar{Y}_m}}}$$

$$= \left(\frac{n+m}{n\bar{X}_n + m\bar{Y}_m}\right)^{n+m} \bar{X}_n^{\,n} \, \bar{Y}_m^{\,m}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m} \left(\frac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}\right)^{n} \left(\frac{m\bar{Y}_m}{n\bar{X}_n + m\bar{Y}_m}\right)^{m}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m} \left(\frac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}\right)^{n} \left(1 - \frac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}\right)^{m}.$$

The LRT rejects H_0 when

$$\frac{(n+m)^{n+m}}{n^n m^m} \left(\frac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}\right)^{n} \left(1 - \frac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}\right)^{m} < k$$

for some k.

(b) Note that the LRT depends on $\dfrac{n\bar{X}_n}{n\bar{X}_n + m\bar{Y}_m}$.

(c) Under $H_0: \theta = \mu$, we have

$$n\bar{X}_n \sim \text{Gamma}(n, \theta) \quad \perp\!\!\!\perp \quad m\bar{Y}_m \sim \text{Gamma}(m, \theta).$$

Exercise 4.19 (b) [Assigned in STAT 412, hw 6], gives

$$\frac{n\bar{x}_n/\theta}{n\bar{x}_n/\theta + m\bar{y}_m/\theta} \sim \text{Beta}(n, m).$$

$\boxed{8.31}$ Let $X_1, \dots, X_n \overset{ind}{\sim} \text{Poisson}(\lambda)$. Test $H_0: \lambda \leq \lambda_0$ vs $H_1: \lambda > \lambda_0$.

(a) For $\lambda_1 > \lambda_0$, write

$$\frac{L(\lambda_0; \underline{X})}{L(\lambda_1; \underline{X})} = \frac{e^{-n\lambda_0} \lambda_0^{n\bar{x}_n} / (\prod_{i=1}^n x_i!)}{e^{-n\lambda_1} \lambda_1^{n\bar{x}_n} / (\prod_{i=1}^n x_i!)}$$

$$= e^{-n(\lambda_0 - \lambda_1)} \underbrace{\left(\frac{\lambda_0}{\lambda_1}\right)}_{<1}^{n\bar{x}_n} \leftarrow \text{decreasing in } \bar{x}_n$$

Since this is decreasing in $\bar{x}_n$, the UMP test has rejection rule

Reject H_0 if $\bar{x}_n > c$

for some c.

(b) For testing $H_0: \lambda \leq \lambda_0$ vs $H_1: \lambda > \lambda_0$, at size ≈ 0.05.

Note that $\frac{\sqrt{n}(\bar{x}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{D} N(0, 1)$ as $n \to \infty$.

The power function admits the large-n approximation

$$\gamma_n(\lambda) = P\left(\bar{X}_n > c\right) = P_\lambda\left(\frac{\sqrt{n}\,(\bar{X}_n - \lambda)}{\sqrt{\lambda}} > \frac{\sqrt{n}\,(c-\lambda)}{\sqrt{\lambda}}\right)$$

for large n.

$$\approx P\left(Z > \frac{\sqrt{n}\,c}{\sqrt{\lambda}} - \sqrt{n}\sqrt{\lambda}\right)$$

$$= \gamma_\infty(\lambda)$$

Setting $\gamma_\infty(\lambda=1) = \alpha$ gives

$$P\left(Z > \sqrt{n}\,c - \sqrt{n}\right) = \alpha$$

$$\iff \quad Z_\alpha = \sqrt{n}\,c - \sqrt{n}$$

$$\iff \quad \frac{Z_\alpha}{\sqrt{n}} + 1 = c_\alpha .$$

The power under c_α is approximately

$$\gamma_\infty^\alpha(\lambda) = P\left(Z > \frac{\sqrt{n}\,c_\alpha}{\sqrt{\lambda}} - \sqrt{n}\sqrt{\lambda}\right)$$

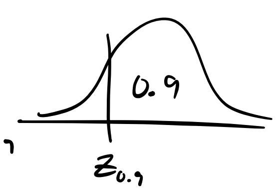
$$= P\left(Z > \frac{\sqrt{n}}{\sqrt{\lambda}}\left(\frac{Z_\alpha}{\sqrt{n}} + 1\right) - \sqrt{n}\sqrt{\lambda}\right)$$

$$= P\left(Z > \frac{Z_\alpha}{\sqrt{\lambda}} + \frac{\sqrt{n}}{\sqrt{\lambda}} - \sqrt{n}\sqrt{\lambda}\right).$$

Choose smallest n such that

$$\gamma_\infty^{0.05}(\lambda=2) \geq 0.9$$

$$\iff \quad P\left(Z > \frac{Z_{0.05}}{\sqrt{2}} + \frac{\sqrt{n}}{\sqrt{2}} - \sqrt{n}\sqrt{2}\right) \geq 0.9$$



$$\iff \quad Z_{.9} \geq \frac{Z_{0.05}}{\sqrt{2}} + \frac{\sqrt{n}}{\sqrt{2}} - \sqrt{n}\sqrt{2}$$

$\Leftrightarrow$

$$-z_{0.10} - \frac{z_{0.05}}{\sqrt{2}} \geq \sqrt{n}\,\Big(\underbrace{\frac{1}{\sqrt{2}} - \sqrt{2}}\Big)$$

$$\frac{1-2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\sqrt{2}\left(z_{0.10} + \frac{z_{0.05}}{\sqrt{2}}\right) \leq \sqrt{n}$$

$$2\left(z_{0.10} + \frac{z_{0.05}}{\sqrt{2}}\right)^2 \leq n$$

$$\underbrace{2\left(1.28 + \frac{1.645}{\sqrt{2}}\right)^2}_{\substack{\shortparallel \\ 11.93}} \leq n$$

so take $n = 12$.