STAT 713 hw 8

Asymptotic tests and interval estimators

Do problems 9.3, 9.4, 9.12, 9.13, 9.17 from CB. In addition:

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Gamma}(2, \beta)$, $\beta > 0$.

   (a) Show that the LRT for H_0: $\beta = \beta_0$ has a rejection rule of the form $\hat{\beta}_n/\beta_0 < c_1$ or $\hat{\beta}_n/\beta_0 > c_2$, where $\hat{\beta}_n$ is the MLE and c_1 and c_2 satisfy $c_1 < c_2$ and $c_1 e^{-c_1} = c_2 e^{-c_2}$.

   Using the fact that the MLE for β is $\hat{\beta}_n = \bar{X}_n/2$, we can write the likelihood ratio as

   $$\text{LR}(\mathbf{X}) = \left[\left(\frac{\hat{\beta}_n}{\beta_0}\right)\exp\left(-\frac{\hat{\beta}_n}{\beta_0}\right)\right]^{2n} e^{2n}.$$

   Noting the shape of the function $z \mapsto ze^{-z}$, we see that there exists $c_1 < c_2$ such that $\text{LR}(\mathbf{X}) < k \iff \hat{\beta}_n/\beta_0 < c_1$ or $\hat{\beta}_n/\beta_0 > c_2$; these values satisfy $c_1 e^{-c_1} = c_2 e^{-c_2}$.

   (b) For $n = 10$, find the values of c_1 and c_2 under which the LRT has size 0.05. You will need to search for these values numerically.

   This code gives one way to find the values numerically.

```
n <- 10
alpha <- 0.05
c1.seq <- seq(qgamma(0.005,2*n,2*n),
              qgamma(0.045,2*n,2*n),
              length = 10000)
c2.seq <- qgamma(pgamma(c1.seq,2*n,2*n) + (1-alpha),2*n, 2*n)
which.c <- which.min(abs(c1.seq*exp(-c1.seq) - c2.seq*exp(-c2.seq)))
c1 <- c1.seq[which.c]
c2 <- c2.seq[which.c]
```

   We obtain $c_1 = 0.6219672$ and $c_2 = 1.5068728$.

   (c) For $n = 10$, compare c_1 and c_2 to the 0.025 and 0.975 quantiles of the distribution of $\hat{\beta}_n/\beta_0$ under H_0: $\beta = \beta_0$. *These "equal tails" critical values are used more commonly in practice than c_1 and c_2 and are much easier to find!*

   We can find these with the code:

```
c1.eq <- qgamma(0.025,2*n,2*n)
c2.eq <- qgamma(0.975,2*n,2*n)
```

   We have $c_1 = 0.610826$ and $c_2 = 1.4835427$, which are very close to the values prescribed by the size-α LRT.

(d) Give the form of the CI for β based on inverting the size-α LRT test of H_0: $\beta = \beta_0$.

We construct the CI by collecting all the values of β_0 for which the LRT will not reject H_0: $\beta = \beta_0$. These values are $\{\beta_0 : c_1 < \hat{\beta}_n/\beta_0 < c_2\} = \{\beta_0 : \hat{\beta}_n/c_2 < \beta_0 < \hat{\beta}_n/c_1\}$, so the CI is given by $[\hat{\beta}_n/c_2, \hat{\beta}_n/c_1]$, where $c_1 < c_2$ are the values chosen as in part (b).

(e) Give the form of the CI for β based on inverting the size-α score test of H_0: $\beta = \beta_0$.

The score function and Fisher information are given by

$$S(\beta; \mathbf{X}) = -\frac{2n}{\beta} + \frac{n\bar{X}_n}{\beta^2} \quad \text{and} \quad I_n(\beta) = \frac{2n}{\beta^2}.$$

The score test test statistic is given by

$$\frac{[S(\beta_0; \mathbf{X})]^2}{I_n(\beta_0)} = \frac{\beta_0^2}{2n}\left[-\frac{2n}{\beta_0} + \frac{n\bar{X}_n}{\beta_0^2}\right]^2 = 2n[1 - \hat{\beta}_n/\beta_0]^2.$$

So the score test rejects H_0: $\beta = \beta_0$ when

$$2n[1 - \hat{\beta}_n/\beta_0]^2 > \chi^2_{1,\alpha}.$$

The corresponding confidence interval for β is the set

$$\{\beta_0 : 2n[1 - \hat{\beta}_n/\beta_0]^2 \le \chi^2_{1,\alpha}\}.$$

To find the endpoints of this interval, we find the values of β_0 which solve the equation

$$\left(1 - \frac{\chi^2_{1,\alpha}}{2n}\right) + (-2\hat{\beta}_n)\left(\frac{1}{\beta_0}\right) + \hat{\beta}_n^2\left(\frac{1}{\beta_0}\right)^2 = 0.$$

Using the quadratic formula, we obtain the interval

$$\left(\frac{\hat{\beta}_n}{1 + \sqrt{\chi^2_{1,\alpha}}}, \frac{\hat{\beta}_n}{1 - \sqrt{\chi^2_{1,\alpha}}}\right).$$

(f) Give the form of the CI for β obtained by inverting the cdf of $\sum_{i=1}^n X_i$.

We have $\sum_{i=1}^n X_i \sim \text{Gamma}(2n, \beta)$, so $F_{\text{Gamma}(2n,\beta)}(\sum_{i=1}^n X_i) \sim \text{Uniform}(0, 1)$. In consequence, a $(1 - \alpha) \times 100\%$ CI for β can be obtained as

$$\{\beta : \alpha/2 \le F_{\text{Gamma}(2n,\beta)}(\textstyle\sum_{i=1}^n X_i) \le 1 - \alpha/2\}.$$

(g) Justify the confidence interval $\hat{\beta}_n \pm z_{\alpha/2}\hat{\beta}_n/\sqrt{2n}$. What name would you give it?

This is based on the result $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\text{D}} \text{Normal}(0, \beta^2/2)$ as $n \to \infty$, from which we have $\sqrt{n}(\hat{\beta}_n - \beta)/(\hat{\beta}_n/\sqrt{2}) \xrightarrow{\text{D}} \text{Normal}(0, 1)$, since $\hat{\beta}_n$ is a consistent estimator of β. From here we see that the interval $\hat{\beta}_n \pm z_{\alpha/2}\hat{\beta}_n/\sqrt{2n}$ will contain β with probability approaching $1 - \alpha$ as $n \to \infty$. This is a Wald-type interval.

(h) Construct the 95% confidence intervals from parts (d), (e), (f), and (g) using the data

```
X <- c(0.99, 10.63, 7.70, 5.23, 4.20, 10.74, 2.69, 7.37, 4.51, 9.05)
```

The following code computes the intervals:

```r
X <- c(0.99, 10.63, 7.70, 5.23, 4.20, 10.74, 2.69, 7.37, 4.51, 9.05)
n <- length(X)
beta.hat <- mean(X) / 2
alpha <- 0.05

# search for critical values for the LRT
c1.seq <- seq(qgamma(0.005,2*n,2*n),
          qgamma(0.045,2*n,2*n),
          length = 10000)
c2.seq <- qgamma(pgamma(c1.seq,2*n,2*n) + (1-alpha),2*n, 2*n)
which.c <- which.min(abs(c1.seq*exp(-c1.seq) - c2.seq*exp(-c2.seq)))
c1 <- c1.seq[which.c]
c2 <- c2.seq[which.c]

# CI from inverting the LRT:
LRT.ci <- c(beta.hat/c2,beta.hat/c1)

# CI from inverting the score test
score.ci <- c(beta.hat/(1 + sqrt(qchisq(1-alpha,1)/(2*n))),
          beta.hat/(1 - sqrt(qchisq(1-alpha,1)/(2*n))))

# CI from pivoting the cdf
beta0.seq <- seq(min(X)/2,max(X),length = 5000) # choose reasonable range
which.lo <- which.min(abs(pgamma(sum(X),2*n,1/beta0.seq)-(1-alpha/2)))
which.up <- which.min(abs(pgamma(sum(X),2*n,1/beta0.seq)-alpha/2))
cdf.ci <- c(beta0.seq[which.lo],beta0.seq[which.up])

# Wald CI
Wald.ci <- c(beta.hat - qnorm(1-alpha/2)*beta.hat/sqrt(2*n),
         beta.hat + qnorm(1-alpha/2)*beta.hat/sqrt(2*n))

LRT.ci
## [1] 2.094072 5.073418
score.ci
## [1] 2.193969 5.617380
cdf.ci
## [1] 2.126330 5.165605
Wald.ci
## [1] 1.772567 4.538433
```

(i) Set $\beta = 3$, $n = 10$, and generate 5000 datasets; on each data set record for each of the four intervals i) whether it contained the true value of β and ii) its width. Report the proportion of times each interval contained its target and its average width.

My simulation based on 500 data sets resulted in:

```
summtable
```

```
##            LRT invert score invert cdf pivot  Wald
## coverage        0.946        0.948     0.944 0.930
## avg width       2.802        3.220     2.858 2.601
```

Problem 9.3, 9.4, 9.12 from CB

$\boxed{9.3}$ Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} f_X(x; \alpha_0, \beta) = \frac{\alpha_0}{\beta}\left(\frac{x}{\beta}\right)^{\alpha_0 - 1} \mathbb{1}(0 \le x \le \beta)$, α_0 known, $\beta > 0$ unknown.

(a)

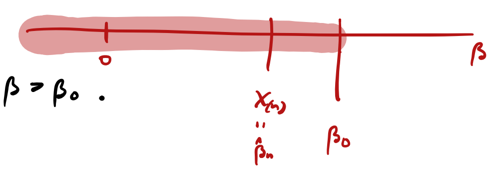
The likelihood function is

$$L(\beta; \underset{\sim}{X}) = \left(\frac{\alpha_0}{\beta}\right)^n \left(\prod_{i=1}^n \frac{X_i}{\beta}\right)^{\alpha_0 - 1} \mathbb{1}(X_{(n)} \le \beta)$$

$$= \underbrace{\alpha_0^n \left(\frac{1}{\beta}\right)^{n\alpha_0}}_{\text{decreasing in } \beta} \left(\prod_{i=1}^n X_i\right)^{\alpha_0 - 1} \mathbb{1}(X_{(n)} \le \beta).$$

We see that

$$\hat{\beta}_n = \underset{\beta > 0}{\operatorname{argmax}}\ L(\beta; \underset{\sim}{X}) = X_{(n)}$$



(i) Consider the LRT for $H_0: \beta \le \beta_0$ vs $H_1: \beta > \beta_0$.

The likelihood ratio has numerator

$$\sup_{\beta \le \beta_0} L(\beta; \underset{\sim}{X}) = \sup_{\beta \le \beta_0} \alpha_0^n \left(\frac{1}{\beta}\right)^{n\alpha_0} \left(\prod_{i=1}^n X_i\right)^{\alpha_0 - 1} \mathbb{1}(\overset{X_{(n)}}{\hat{\beta}_n} \le \beta)$$

$$= \begin{cases} 0 & \text{if } \hat{\beta}_n > \beta_0 \\ \alpha_0^n \left(\frac{1}{\hat{\beta}_n}\right)^{n\alpha_0} \left(\prod_{i=1}^n X_i\right)^{\alpha_0 - 1} \mathbb{1}(\overset{X_{(n)}}{\hat{\beta}_n} \le \hat{\beta}_n) & \text{if } \hat{\beta}_n \le \beta_0, \end{cases}$$

and denominator

$$\sup_{\beta > 0} L(\beta; \underset{\sim}{X}) = \alpha_0^n \left(\frac{1}{\hat{\beta}_n}\right)^{n\alpha_0} \left(\prod_{i=1}^n X_i\right)^{\alpha_0 - 1} \mathbb{1}(\overset{X_{(n)}}{\hat{\beta}_n} \le \hat{\beta}_n),$$

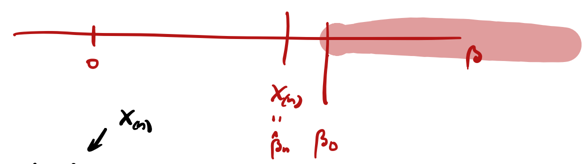
so that

$$LR(\underset{\sim}{X}) = \begin{cases} 0 & \text{if } \hat{\beta}_n > \beta_0 \\ 1 & \text{if } \hat{\beta}_n \le \beta_0. \end{cases}$$

This makes sense — but it is not helpful for building a C.I. for β.

(ii) Consider the LRT of $H_0: \beta \geq \beta_0$ vs $H_1: \beta < \beta_0$.

Then the likelihood ratio has numerator



$$\sup_{\beta \geq \beta_0} L(\beta; \underset{\sim}{X}) = \sup_{\beta \geq \beta_0} \alpha_0^n \left(\frac{1}{\beta}\right)^{n\alpha_0} \left(\prod_{i=1}^n x_i\right)^{\alpha_0 - 1} \mathbb{1}(\underbrace{\hat{\beta}_n}_{X_{(n)}} \leq \beta)$$

$$= \begin{cases} \alpha_0^n \left(\frac{1}{\beta_0}\right)^{n\alpha_0} \left(\prod_{i=1}^n x_i\right)^{\alpha_0 - 1} \mathbb{1}(\hat{\beta}_n \leq \beta_0) & \text{if } \hat{\beta}_n < \beta_0 \\ \alpha_0^n \left(\frac{1}{\hat{\beta}_n}\right)^{n\alpha_0} \left(\prod_{i=1}^n x_i\right)^{\alpha_0 - 1} \mathbb{1}(\hat{\beta}_n \leq \hat{\beta}_n) & \text{if } \hat{\beta}_n \geq \beta_0 . \end{cases}$$

The denominator is the same as for the previous hypotheses, so we have

$$LR(\underset{\sim}{X}) = \begin{cases} \left(\frac{\hat{\beta}_n}{\beta_0}\right)^{n\alpha_0} & \text{if } \hat{\beta}_n < \beta_0 \\ 1 & \text{if } \hat{\beta}_n \geq \beta_0 . \end{cases}$$

The LRT rejects $H_0: \beta \geq \beta_0$ in favor of $H_1: \beta < \beta_0$ if

$$\left(\frac{\hat{\beta}_n}{\beta_0}\right)^{n\alpha_0} < k.$$

We now calibrate k to give the test size α: Then we can invert the test to obtain a confidence interval for β.

The MLE $\hat{\beta}_n = X_{(n)}$ has cdf

$$F_{X_{(n)}}(x; \beta) = \left[F_X(x;\beta)\right]^n = \left[\left(\frac{x}{\beta}\right)^{\alpha_0}\right]^n = \left(\frac{x}{\beta}\right)^{n\alpha_0}, \quad \text{for } x \in [0, \beta].$$

This allows us to write the power function of the LRT as

$$\begin{aligned}
\gamma(\beta) &= P_\beta\left(\left(\frac{\hat{\beta}_n}{\beta_0}\right)^{n\alpha_0} < k\right) \\
&= P_\beta\left(\hat{\beta}_n < \beta_0\, k^{\frac{1}{n\alpha_0}}\right) \\
&= \left(\frac{\beta_0\, k^{\frac{1}{n\alpha_0}}}{\beta}\right)^{n\alpha_0} \\
&= \left(\frac{\beta_0}{\beta}\right)^{n\alpha_0} \cdot k
\end{aligned}$$

The size of the test is

$$\sup_{\beta \geq \beta_0} \gamma(\beta) = \gamma(\beta_0) = \left(\frac{\beta_0}{\beta_0}\right)^{n\alpha_0} \cdot k = k,$$

so setting $k = \alpha$ gives a size α test.

In summary, the size-α LRT for $H_0: \beta \geq \beta_0$ vs $H_1: \beta < \beta_0$ rejects H_0 when

$$\left(\frac{\hat{\beta}_n}{\beta_0}\right)^{n\alpha_0} < \alpha.$$

Inverting this test to obtain a C.I. for β gives the interval

$$\begin{aligned}
\left\{\beta_0 : \left(\frac{\hat{\beta}_n}{\beta_0}\right)^{n\alpha_0} < \alpha\right\} &= \left\{\beta_0 : \hat{\beta}_n < \alpha^{\frac{1}{n\alpha_0}}\beta_0\right\} \\
&= \left(\hat{\beta}_n\, \alpha^{-\frac{1}{n\alpha_0}}, \infty\right).
\end{aligned}$$

(b) If α_0 is unknown, the MLE is

$$\hat{\alpha} = \underset{\alpha>0}{\operatorname{argmax}} \left(\frac{\alpha}{\hat{\beta}_n}\right)^n \left(\prod_{i=1}^n \frac{X_i}{\hat{\beta}_n}\right)^{\alpha-1} \underbrace{\mathbb{1}(X_{(n)} \le \hat{\beta}_n)}_{=1}$$

$$= \underset{\alpha>0}{\operatorname{argmax}} \left\{ n \log \alpha + (\alpha-1)\sum_{i=1}^n \left(\log X_i - \log \hat{\beta}_n\right)\right\}.$$

We have

$$\frac{\partial}{\partial \alpha}\left\{ n \log \alpha + (\alpha-1)\sum_{i=1}^n \left(\log X_i - \log \hat{\beta}_n\right)\right\} = \frac{n}{\alpha} + \sum_{i=1}^n \left(\log X_i - \log X_{(n)}\right) \overset{\text{set}}{=} 0$$

$$\iff \hat{\alpha} = \frac{1}{\frac{1}{n}\sum_{i=1}^n \left[\log X_{(n)} - \log X_i\right]}$$

On the data

22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7, 23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0,

we obtain $\hat{\alpha} = 12.59487$, $\hat{\beta}_n = 25$.

The 95% interval is $(25.4287, \infty)$.

$\boxed{9.4}$ Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma_X^2) \perp\!\!\!\perp Y_1, \dots, Y_m \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma_Y^2)$.

Let $\lambda = \sigma_X^2 / \sigma_Y^2$ [Oops, book problem uses $\lambda = \sigma_Y^2/\sigma_X^2$]

(a)(b) Consider the LRT of $H_0: \lambda = \lambda_0$ vs $H_1: \lambda \neq \lambda_0$.

The likelihood is given by

$$\mathcal{L}(\sigma_X^2, \sigma_Y^2; \underset{\sim}{X}, \underset{\sim}{Y}) = (2\pi)^{-n/2}(\sigma_X^2)^{-n/2}\exp\left[-\frac{\sum_{i=1}^n X_i^2}{2\sigma_X^2}\right](2\pi)^{-m/2}(\sigma_Y^2)^{-m/2}\exp\left[-\frac{\sum_{i=1}^m Y_i^2}{2\sigma_Y^2}\right].$$

The mles of σ_X^2 and σ_Y^2 are $\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$, $\hat{\sigma}_Y^2 = \frac{1}{m}\sum_{j=1}^m Y_j$

Under $H_0: \dfrac{\sigma_X^2}{\sigma_Y^2} = \lambda_0 \iff \sigma_X^2 = \lambda_0 \sigma_Y^2$, the restricted mles are:

$$\left(\hat{\sigma}^2_{X,0}, \hat{\sigma}^2_{Y,0}\right) = \left(\lambda_0 \hat{\sigma}^2, \hat{\sigma}^2\right),$$

where

$$\hat{\sigma}^2 = \underset{\sigma^2 > 0}{\operatorname{argmax}} \; (2\pi)^{-n/2} (\lambda_0 \sigma^2)^{-n/2} \exp\left[-\frac{\sum_{i=1}^n X_i^2}{2\lambda_0 \sigma^2}\right] (2\pi)^{-m/2} (\sigma^2)^{-m/2} \exp\left[-\frac{\sum_{i=1}^m Y_i^2}{2\sigma^2}\right].$$

We write

$$\ell(\lambda\sigma^2, \sigma^2; \underset{\sim}{X}, \underset{\sim}{Y}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\lambda_0 - \frac{n}{2}\log\sigma^2 - \frac{1}{2\lambda_0\sigma^2}\sum_{i=1}^n X_i^2$$

$$-\frac{m}{2}\log(2\pi) - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{j=1}^m Y_j^2 .$$

$$\frac{\partial}{\partial\sigma^2}\ell(\lambda\sigma^2, \sigma^2; \underset{\sim}{X}, \underset{\sim}{Y}) = -\frac{n}{2\sigma^2} - \frac{m}{2\sigma^2} + \frac{1}{2\lambda_0\sigma^4}\sum_{i=1}^n X_i^2 + \frac{1}{2\sigma^4}\sum_{j=1}^m Y_j \overset{\text{set}}{=} 0.$$

$$\iff \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^n X_i^2/\lambda_0 + \sum_{j=1}^m Y_j}{n+m} = \frac{n\hat{\sigma}_X^2/\lambda_0 + m\hat{\sigma}_Y^2}{n+m}$$

So the likelihood ratio is given by

$$LR(\underset{\sim}{X}, \underset{\sim}{Y}) = \frac{(2\pi)^{-n/2}(\lambda_0\hat{\sigma}^2)^{-n/2}\exp\left[-\frac{\sum_{i=1}^n X_i^2}{2\lambda_0\hat{\sigma}^2}\right](2\pi)^{-m/2}(\hat{\sigma}^2)^{-m/2}\exp\left[-\frac{\sum_{i=1}^m Y_i^2}{2\hat{\sigma}^2}\right]}{(2\pi)^{-n/2}(\hat{\sigma}_X^2)^{-n/2}\exp\left[-\frac{\sum_{i=1}^n X_i^2}{2\hat{\sigma}_X^2}\right](2\pi)^{-m/2}(\hat{\sigma}_Y^2)^{-m/2}\exp\left[-\frac{\sum_{i=1}^m Y_i^2}{2\hat{\sigma}_Y^2}\right]}$$

$$= \left(\frac{\hat{\sigma}_X^2}{\lambda_0\hat{\sigma}^2}\right)^{n/2}\left(\frac{\hat{\sigma}_Y^2}{\hat{\sigma}^2}\right)^{m/2}$$

$$= \left(\frac{\hat\sigma_X^2/\lambda_0}{\left[\frac{n\hat\sigma_X^2/\lambda_0 + m\hat\sigma_Y^2}{n+m}\right]} \right)^{n/2} \left(\frac{\hat\sigma_Y^2}{\frac{n\hat\sigma_X^2/\lambda_0 + m\hat\sigma_Y^2}{n+m}} \right)^{m/2}$$

$$= \frac{(n+m)^{\frac{n+m}{2}}}{n^{n/2}\, m^{m/2}} \left(\frac{n\hat\sigma_X^2/\lambda_0}{n\hat\sigma_X^2/\lambda_0 + m\hat\sigma_Y^2} \right)^{n/2} \left(\frac{m\hat\sigma_Y^2}{n\hat\sigma_X^2/\lambda_0 + m\hat\sigma_Y^2} \right)^{m/2}$$

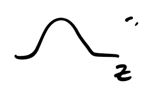
$$= \frac{(n+m)^{\frac{n+m}{2}}}{n^{n/2}\, m^{m/2}} \left(\frac{\left[\frac{n\hat\sigma_X^2}{m\hat\sigma_Y^2}/\lambda_0\right]}{\left[\frac{n\hat\sigma_X^2}{m\hat\sigma_Y^2}/\lambda_0\right] + 1} \right)^{n/2} \left(\frac{1}{\left[\frac{n\hat\sigma_X^2}{m\hat\sigma_Y^2}/\lambda_0\right] + 1} \right)^{m/2}$$

$$= \frac{(n+m)^{\frac{n+m}{2}}}{n^{n/2}\, m^{m/2}} \left(\frac{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right]}{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] + \frac{m}{n}} \right)^{n/2} \left(\frac{\frac{m}{n}}{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] + \frac{m}{n}} \right)^{m/2}$$

so

$$LR(\underline{X}, \underline{Y}) < k$$

$$\iff \left(\frac{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right]}{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] + \frac{m}{n}} \right)^{n} \left(\frac{\frac{m}{n}}{\left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] + \frac{m}{n}} \right)^{m} < k^* .$$

The function $\left(\frac{z}{z+\frac{m}{n}}\right)^n \left(\frac{\frac{m}{n}}{z+\frac{m}{n}}\right)^m$ has the shape "  ", so

$$LR(\underline{X}, \underline{Y}) < k$$

$$\iff \left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] < c_1 \quad \text{or} \quad \left[\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}/\lambda_0\right] > c_2 .$$

We must find c_1 and c_2 such that the LRT has size α.

We begin by writing the power function: Recalling that $\lambda = \frac{\sigma_X^2}{\sigma_Y^2}$, we have

$$\gamma(\lambda) = P_\lambda\left(\frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \Big/ \lambda_0 \notin (c_1, c_2) \right)$$

$$= P_\lambda\left(\underbrace{\frac{\hat\sigma_X^2 / \sigma_X^2}{\hat\sigma_Y^2 / \sigma_Y^2}}_{\sim F_{n,m}} \notin (\lambda_0 c_1, \lambda_0 c_2) \right)$$

$$= P_\lambda\left(\frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \Big/ \lambda \notin \left(\frac{\lambda_0}{\lambda} c_1, \frac{\lambda_0}{\lambda} c_2 \right) \right)$$

$$= P\left(R \notin \left(\frac{\lambda_0}{\lambda} c_1, \frac{\lambda_0}{\lambda} c_2 \right) \right), \qquad R \sim F_{n,m}$$

The size of the test is given by

$$\gamma(\lambda_0) = P\left(R \notin (c_1, c_2) \right), \qquad R \sim F_{n,m}.$$

So we must find $c_1 < c_2$ such that

$$P\left(R \in (c_1, c_2) \right) = 1 - \alpha$$

and

$$\left(\frac{c_1}{c_1 + \frac{m}{n}} \right)^n \left(\frac{\frac{m}{n}}{c_1 + \frac{m}{n}} \right)^m = \left(\frac{c_2}{c_2 + \frac{m}{n}} \right)^n \left(\frac{\frac{m}{n}}{c_2 + \frac{m}{n}} \right)^m .$$

Example code for finding c_1 and c_2

```
n <- 10
m <- 8

c1.seq <- seq(qf(0.005,n,m),qf(0.045,n,m),by = 0.001)
c2.seq <- qf(pf(c1.seq,n,m) + 0.95,n,m)
f1 <- (c1.seq/(c1.seq + m/n))^n*((m/n)/(c1.seq + m/n))^m
f2 <- (c2.seq/(c2.seq + m/n))^n*((m/n)/(c2.seq + m/n))^m
which.c <- which.min(abs(f1 - f2))
c1 <- c1.seq[which.c]
c2 <- c2.seq[which.c]

pf(c1,n,m)
pf(c2,n,m)
```

An alternative to finding c_1 and c_2 which exactly correspond to the LRT is to simply take

$$c_1 = F_{n,m,1-\alpha/2} \quad \text{and} \quad c_2 = F_{n,m,\alpha/2} .$$

You will still obtain a size-α test (it just will not be exactly the LRT).

(c) Having found c_1 and c_2, a $(1-\alpha)^*100\%$ C.I. for λ is

$$\left\{ \lambda_0 : \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \Big/ \lambda_0 \in [c_1, c_2] \right\}$$

$$= \left\{ \lambda_0 : \frac{\lambda_0}{\frac{\hat\sigma_X^2}{\hat\sigma_Y^2}} \in \left[\frac{1}{c_2}, \frac{1}{c_1} \right] \right\}$$

$$= \left\{ \lambda_0 : \lambda_0 \in \left[\frac{1}{c_2} \cdot \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} , \frac{1}{c_1} \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \right] \right\} .$$

So the $(1-\alpha)^*100\%$ C.I. is

$$\left[\frac{1}{c_2} \cdot \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} , \frac{1}{c_1} \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \right] .$$

Under $c_1 = F_{n,m,1-\alpha/2}$ and $c_2 = F_{n,m,\alpha/2}$ this would be

$$\left[\frac{1}{F_{n,m,\alpha/2}} \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} , \frac{1}{F_{n,m,1-\alpha/2}} \frac{\hat\sigma_X^2}{\hat\sigma_Y^2} \right] .$$

$\boxed{9.12}$ Let $X_1, \dots, X_n \overset{ind}{\sim} \text{Normal}(\theta, \theta)$, $\theta > 0$.

Then

$$Q(\underset{\sim}{X}; \theta) = \frac{\sqrt{n}\,(\bar{X}_n - \theta)}{\sqrt{\theta}}$$

is a pivotal quantity with the Normal$(0,1)$ distribution.

We have $Q(\underset{\sim}{X}; \theta) \in [-Z_{\alpha/2}, Z_{\alpha/2}]$ with probability $1-\alpha$.

So a $(1-\alpha)^* 100\%$ C.I. for θ is

$$\left\{\theta : \frac{\sqrt{n}\,(\bar{X}_n - \theta)}{\sqrt{\theta}} \in [-Z_{\alpha/2}, Z_{\alpha/2}]\right\}$$

$$= \left\{\theta : n(\bar{X}_n - \theta)^2 < \chi^2_{1,\alpha} \cdot \theta\right\}$$

$$= \left\{\theta : n\bar{X}_n^2 - n2\bar{X}_n\theta + n\theta^2 < \chi^2_{1,\alpha} \cdot \theta\right\}$$

$$= \left\{\theta : n\bar{X}_n^2 - \left(2n\bar{X}_n + \chi^2_{1,\alpha}\right)\theta + n\theta^2 < 0\right\}$$

By the quadratic formula, this is the interval

$$\left[\frac{2n\bar{X}_n + \chi^2_{1,\alpha} \pm \sqrt{\left(2n\bar{X}_n + \chi^2_{1,\alpha}\right)^2 - 4(n\bar{X}_n)^2}}{2n}\right]$$

$$= \left[\bar{X}_n + \frac{\chi^2_{1,\alpha}}{2n} \pm \sqrt{\left(\bar{X}_n + \frac{\chi^2_{1,\alpha}}{2n}\right)^2 - \bar{X}_n^2}\right]$$