

STAT 713 sp 2022 Exam 1

1. Consider the hierarchical model

$$X_1, \dots, X_n | \theta \stackrel{\text{ind}}{\sim} p(x; \theta) = \theta(1 - \theta)^{x-1} \mathbf{1}(x \in \{1, 2, \dots\})$$
$$\theta \sim \pi(\theta; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1 - \theta)^{b-1} \mathbf{1}(0 < \theta < 1),$$

for some prior parameters $a > 0$ and $b > 0$.

- (a) Give the Bayesian estimator $\hat{\theta}_{\text{Bayes}} := \mathbb{E}[\theta | X_1, \dots, X_n]$ in terms of X_1, \dots, X_n and a and b .
- (b) Give the MLE $\hat{\theta}_{\text{MLE}}$ of θ based on X_1, \dots, X_n .
- (c) Describe what happens to $|\hat{\theta}_{\text{Bayes}} - \hat{\theta}_{\text{MLE}}|$ as $n \rightarrow \infty$.

2. Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f(x; \theta) = \theta x^{-(\theta+1)} \mathbf{1}(x > 1)$.

(a) Find a complete sufficient statistic for θ .

(b) Find the MLE for θ .

(c) Find the UMVUE for $\tau = \tau(\theta) = 1/\theta$. Prove that your estimator is the UMVUE.

(d) Propose an unbiased estimator, say $\tilde{\eta}$, of $\eta = \eta(\theta) = P_\theta(X_1 \leq a) = 1 - a^{-\theta}$, for some $a > 1$. Then describe how you could find the UMVUE $\hat{\eta}$ for η (you do not need to carry out the steps).

3. Let $X_1, \dots, X_n \stackrel{\text{ind}}{\sim} f(x; \theta) = \frac{1}{\theta} e^{-(x-\theta)/\theta} \cdot \mathbf{1}(x > \theta)$, where $\theta > 0$.

(a) Find a statistic which is ancillary to θ .

(b) Find a minimal sufficient statistic for θ .

(c) Check whether your minimal sufficient statistic is a complete statistic. *Hint: Can you find a function of your statistic which has expectation zero—but which is not identically zero?*