STAT 713 sp 2022 Exam 1

1. Consider the hierarchical model

$$X_1, \dots, X_n | \theta \overset{\text{ind}}{\sim} p(x; \theta) = \theta(1-\theta)^{x-1}\mathbf{1}(x \in \{1, 2, \dots\})$$
$$\theta \sim \pi(\theta; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}\mathbf{1}(0 < \theta < 1),$$

for some prior parameters $a > 0$ and $b > 0$.

(a) Give the Bayesian estimator $\hat{\theta}_{\text{Bayes}} := \mathbb{E}[\theta | X_1, \dots, X_n]$ in terms of $X_1, \dots, X_n$ and a and b.

To find the posterior distribution of θ conditional on $X_1, \dots, X_n$, we write

$$\begin{aligned}\pi(\theta | X_1, \dots, X_n) &\propto \theta^n (1-\theta)^{n\bar{X}_n - n} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \\ &\propto \theta^{(n+a)-1}(1-\theta)^{n\bar{X}_n - n + b}.\end{aligned}$$

Note that this is proportional to the pdf of the Beta$(n + a, n\bar{X}_n - n + b)$ distribution, so

$$\theta | X_1, \dots, X_n \sim \text{Beta}(n + a, n\bar{X}_n - n + b).$$

The posterior mean of θ given $X_1, \dots, X_n$ is given by

$$\hat{\theta}_{\text{Bayes}} = \frac{n+a}{n\bar{X}_n + a + b}$$

(b) Give the MLE $\hat{\theta}_{\text{MLE}}$ of θ based on $X_1, \dots, X_n$.

The log-likelihood for θ is given by

$$\ell(\theta; X_1, \dots, X_n) = n \log \theta + (n\bar{X}_n - n) \log(1-\theta).$$

Taking the derivative of this with respect to θ and setting it equal to zero gives the MLE

$$\hat{\theta}_{\text{MLE}} = \frac{1}{\bar{X}_n}.$$

(c) Describe what happens to $|\hat{\theta}_{\text{Bayes}} - \hat{\theta}_{\text{MLE}}|$ as $n \to \infty$.

Note that

$$\hat{\theta}_{\text{Bayes}} = \frac{1 + a/n}{\bar{X}_n + (a+b)/n} \approx \frac{1}{\bar{X}_n}$$

for large n. So $|\hat{\theta}_{\text{Bayes}} - \hat{\theta}_{\text{MLE}}| \to 0$ as $n \to \infty$.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\theta) = \theta x^{-(\theta+1)}\mathbf{1}(x > 1)$.

(a) Find a complete sufficient statistic for θ.

This family of distributions is an exponential family, as we can see by expressing the pdf as

$$f(x;\theta) = \theta \cdot \mathbf{1}(x > 1) \cdot \exp\left[-(\theta+1)\cdot \log x\right] = c(\theta)h(x)\exp[w(\theta)t(x)],$$

where $c(\theta) = \theta$, $h(x) = \mathbf{1}(x > 1)$, $w(\theta) = -(\theta+1)$, and $t(x) = \log x$. From here, we can say that $T = \sum_{i=1}^n \log X_i$ is a complete sufficient statistic for θ.

(b) Find the MLE for θ.

The log-likelihood function for θ is given by

$$\ell(\theta; \mathbf{X}) = n\log\theta - (\theta+1)\sum_{i=1}^n \log X_i.$$

Maximizing this in θ (setting the first derivative with respect to θ equal to zero and solving) gives the MLE

$$\hat{\theta}_{\text{MLE}} = \frac{n}{\sum_{i=1}^n \log X_i}$$

for θ.

(c) Find the UMVUE for $\tau = \tau(\theta) = 1/\theta$. Prove that your estimator is the UMVUE.

Let's check whether the MLE for $\tau(\theta) = 1/\theta$ is the UMVUE. We have

$$\hat{\tau}_{\text{MLE}} = \frac{1}{\hat{\theta}_{\text{MLE}}} = \frac{1}{n}\sum_{i=1}^n \log X_i.$$

We may find the density of $Y = \log X_1$ using the transformation method; it is given by $f_Y(y) = \theta e^{-y\theta}\mathbf{1}(y > 0)$, from which we see that $\mathbb{E}Y = 1/\theta = \tau$. Therefore

$$\mathbb{E}\hat{\tau}_{\text{MLE}} = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \log X_i\right] = \mathbb{E}\log X_1 = \tau,$$

that is, $\hat{\tau}_{\text{MLE}}$ is an unbiased estimator of τ. Moreover, it is a function of the complete sufficient statistic $T(\mathbf{X}) = \sum_{i=1}^n \log X_i$, so it is the UMVUE.

(d) Propose an unbiased estimator, say $\tilde{\eta}$, of $\eta = \eta(\theta) = P_\theta(X_1 \le a) = 1 - a^{-\theta}$, for some $a > 1$. Then describe how you could find the UMVUE $\hat{\eta}$ for η (you do not need to carry out the steps).

An unbiased estimator of $\eta(\theta)$ is $\tilde{\eta} = \mathbf{1}(X_1 \le a)$. We may obtain the UMVUE of η as the conditional expectation $\hat{\eta} = \mathbb{E}[\tilde{\eta}|T]$, where $T = \sum_{i=1}^n \log X_i$, since this is a complete sufficient statistic.

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\theta) = \frac{1}{\theta} e^{-(x-\theta)/\theta} \cdot \mathbf{1}(x > \theta)$, where $\theta > 0$.

(a) Find a statistic which is ancillary to θ.

> We can regard this density as belonging to either a location-scale family in which θ is both the location and the scale parameter or a scale family with scale parameter θ. Because of this, any statistic which is location-scale invariant or just scale invariant will be ancillary to θ. We could choose, for example, the statistic $(X_1 - X_2)/(X_3 - X_4)$ or the statistic $X_{(1)}/X_{(n)}$.

(b) Find a minimal sufficient statistic for θ.

> We begin by writing the joint density of $\mathbf{X} = (X_1, \dots, X_n)$ as
>
> $$f(\mathbf{x};\theta) = \theta^{-n} e^{-n\bar{x}_n/\theta} e^n \cdot \mathbf{1}(x_{(1)} > \theta).$$
>
> By the factorization theorem, we see from this expression that $T(\mathbf{X}) = (X_{(1)}, \bar{X}_n)$ is a sufficient statistic for θ. To check whether it is a minimal sufficient statistic, we let $\mathbf{x}$ and $\mathbf{y}$ be any points in $\mathcal{X} = (\theta, \infty)^n$ and write
>
> $$\frac{f(\mathbf{x};\theta)}{f(\mathbf{y};\theta)} = \frac{\theta^{-n} e^{-n\bar{x}_n/\theta} e^n \cdot \mathbf{1}(x_{(1)} > \theta)}{\theta^{-n} e^{-n\bar{y}_n/\theta} e^n \cdot \mathbf{1}(y_{(1)} > \theta)} = \frac{e^{-n\bar{x}_n/\theta} \cdot \mathbf{1}(x_{(1)} > \theta)}{e^{-n\bar{y}_n/\theta} \cdot \mathbf{1}(y_{(1)} > \theta)}.$$
>
> Since this ratio is a constant function of θ if and only if $(x_{(1)}, \bar{x}_n) = (y_{(1)}, \bar{y}_n)$, we have that $T(\mathbf{X}) = (X_{(1)}, \bar{X}_n)$ is a sufficient statistic.

(c) Check whether your minimal sufficient statistic is a complete statistic. *Hint: Can you find a function of your statistic which has expectation zero—but which is not identically zero?*

> If we can find a function $g(X_{(1)}, \bar{X}_n)$ which has expectation equal to zero, then we can conclude that $T(X_{(1)}, \bar{X}_n)$ is not a complete statistic, i.e. the family of pdfs is not complete. We can indeed find such a function $g(X_{(1)}, \bar{X}_n)$ by finding the pdf of $X_{(1)}$, which is
>
> $$f_{X_{(1)}}(x) = \frac{1}{\theta/n} e^{-(x-\theta)/(\theta/n)} \mathbf{1}(x > \theta),$$
>
> from which we can see that $\mathbb{E}X_{(1)} = \theta(n+1)/n$. Using in addition the fact that $\mathbb{E}\bar{X}_n = 2\theta$, we see that the function
>
> $$g(X_{(1)}, \bar{X}_n) = \frac{n}{n+1} X_{(1)} + \frac{1}{2} \bar{X}_n$$
>
> has expectation zero, though it is not equal to zero with unit probability. We conclude that $T(X_{(1)}, \bar{X}_n)$ is not a complete statistic.