STAT 713 sp 2022 Exam 2

1. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} f_X(x;\theta) = \dfrac{2x}{\theta} \exp\left[-\dfrac{x^2}{\theta}\right] \mathbf{1}(x > 0)$ for some $\theta \in (0, \infty)$.
   (a) Give the Cramér-Rao lower bound for unbiased estimators of θ based on $X_1, \ldots, X_n$.
   (b) Give the limiting distribution of $\sqrt{n}(\sqrt{\hat{\theta}_n} - \sqrt{\theta})$ as $n \to \infty$, where $\hat{\theta}_n$ is the MLE for θ.
   (c) Find the method of moments estimator for θ and establish whether it is (weakly) consistent.

2. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} f_X(x;\beta) = \beta x^{-(\beta+1)}\mathbf{1}(x > 1)$, for some $\beta > 0$ and consider testing H_0: $\beta = \beta_0$ versus H_1: $\beta = \beta_1$, where $\beta_1 > \beta_0$.

   (a) Give a test which is the most powerful test among all tests with equal or smaller size.

   (b) Identify a sufficient statistic T for β and determine whether the UMP test rejects when $T > c$ or when $T < c$ for some c.

   (c) Now choose c such that the test has size α. *Hint: You can find the distribution of* $\log X_1$.

3. Let X be a random variable with distribution determined by the hierarchical model

$$\begin{aligned} X|Z &\sim \text{Normal}(0, Z + (1-Z)\pi^2) \\ Z &\sim \text{Bernoulli}(\delta), \end{aligned}$$

for some $\delta \in [0,1]$. Consider testing H_0: $\delta = 1$ versus H_1: $\delta < 1$ with a single realization of X.

(a) Give an expression for the cdf $F_X(x;\delta) = P_\delta(X \leq x)$.
Hint: $\{X \leq x\} = \{X \leq x \cap Z = 0\} \cup \{X \leq x \cap Z = 1\}$.

(b) Give the power function of the test which rejects H_0 when $|X| > c$ for some $c > 0$.

(c) Find the value of c such that the test has size α.