# STAT 713 sp 2022 Exam 2

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x;\theta) = \dfrac{2x}{\theta} \exp\left[-\dfrac{x^2}{\theta}\right] \mathbf{1}(x > 0)$ for some $\theta \in (0, \infty)$.

   (a) Give the Cramér-Rao lower bound for unbiased estimators of $\theta$ based on $X_1, \dots, X_n$.

   The log-likelihood is given by

   $$\ell(\theta; \mathbf{X}) = n \log 2 + \sum_{i=1}^n \log X_i - n \log \theta - \frac{1}{\theta} \sum_{i=1}^n X_i^2,$$

   and the score function by

   $$S(\theta; \mathbf{X}) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n X_i^2.$$

   Noting that $\mathbb{E} X_1^2 = \theta$, which we can obtain from the fact $\mathbb{E}_\theta[S(\theta; \mathbf{X})] = 0$ or by computing an integral, the Fisher information is given by

   $$I_n(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial \theta^2} \ell(\theta; \mathbf{X})\right] = -\mathbb{E}\left[\frac{n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^n X_i^2\right] = \frac{n}{\theta^2}.$$

   The CRLB for unbiased estimators of $\theta$ is the reciprocal of the Fisher information, which is

   $$\frac{1}{I_n(\theta)} = \frac{\theta^2}{n}.$$

   (b) Give the limiting distribution of $\sqrt{n}(\sqrt{\hat{\theta}_n} - \sqrt{\theta})$ as $n \to \infty$, where $\hat{\theta}_n$ is the MLE for $\theta$.

   The Fisher information based on a sample of size 1 is given by $I_1(\theta) = 1/\theta^2$, so we have

   $$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{D}} \text{Normal}(0, \theta^2)$$

   as $n \to \infty$, where the asymptotic variance is obtained as $1/I_1(\theta) = \theta^2$. We now apply the delta method, noting that $\sqrt{\theta} = 1/(2\sqrt{\theta})$. We obtain

   $$\sqrt{n}(\sqrt{\hat{\theta}_n} - \sqrt{\theta}) \xrightarrow{\text{D}} \text{Normal}(0, \theta/4)$$

   as $n \to \infty$, where the asymptotic variance is obtained as $(1/(2\sqrt{\theta}))^2 \theta^2 = \theta/4$.

(c) Find the method of moments estimator for $\theta$ and establish whether it is (weakly) consistent.

We first find

$$\begin{aligned}
m_1 &= \int_0^\infty x \cdot \frac{2x}{\theta} \exp\left[-\frac{x^2}{\theta}\right] dx \\
&= \int_0^\infty \frac{2(\sqrt{\theta y})^2}{\theta} e^{-y} \frac{1}{2}\sqrt{\frac{\theta}{y}} dy \quad \left( y = -x^2/\theta, \quad x = \sqrt{\theta y}, \quad dx = \frac{1}{2}\sqrt{\frac{\theta}{y}} dy \right) \\
&= \sqrt{\theta} \int_0^\infty y^{3/2-1} e^{-y} dy \\
&= \sqrt{\theta}(1/2)\Gamma(1/2) \\
&= \frac{\sqrt{\theta}}{2}\sqrt{\pi},
\end{aligned}$$

which gives the method of moments estimator

$$\tilde{\theta}_n = \frac{4\hat{m}_1^2}{\pi},$$

where $\hat{m}_1 = \bar{X}_n$.

By the WLLN, we have $\hat{m}_1 \xrightarrow{p} m_1$ as $n \to \infty$ provided $m_2 < \infty$, which holds since

$$m_2 = \int_0^\infty x^2 \cdot \frac{2x}{\theta} \exp\left[-\frac{x^2}{\theta}\right] dx = \cdots = \theta < \infty.$$

Now, since the function $g(z) = 4z^2/\pi$ is continuous,

$$\hat{m}_1 \xrightarrow{p} m_1 \quad \implies \quad \tilde{\theta}_n = g(\hat{m}_1) \xrightarrow{p} g(m_1) = \theta$$

as $n \to \infty$.

We could alternatively establish the weak consistency of $\tilde{\theta}_n$ by showing that its bias and variance go to zero as $n \to \infty$. However, $\operatorname{Var} \tilde{\theta}_n$ involves $m_4$, which takes some additional work to obtain.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x;\beta) = \beta x^{-(\beta+1)}\mathbf{1}(x > 1)$, for some $\beta > 0$ and consider testing $H_0\colon \beta = \beta_0$ versus $H_1\colon \beta = \beta_1$, where $\beta_1 > \beta_0$.

(a) Give a test which is the most powerful test among all tests with equal or smaller size.

The UMP test rejects $H_0$ if and only if

$$\frac{\beta_0^n \left(\prod_{i=1}^n \frac{1}{X_i}\right)^{\beta_0+1}}{\beta_1^n \left(\prod_{i=1}^n \frac{1}{X_i}\right)^{\beta_1+1}} < k$$

for some $k$.

(b) Identify a sufficient statistic $T$ for $\beta$ and determine whether the UMP test rejects when $T > c$ or when $T < c$ for some $c$.

The rejection rule of the UMP test is equivalent to

$$\left(\frac{\beta_0}{\beta_1}\right)^n \left(\prod_{i=1}^n X_i\right)^{\beta_1-\beta_0} < k.$$

Since $\beta_1 > \beta_0$, the left side is an increasing function of the statistic $T = \prod_{i=1}^n X_i$, which is sufficient for $\beta$ by the factorization theorem. Since the UMP test rejects $H_0$ when the ratio of likelihoods is small, it is equivalent to the test which rejects $H_0$ when the statistic $T$ is small, that is when $T < c$ for some $c$.

(c) Now choose $c$ such that the test has size $\alpha$. *Hint: You can find the distribution of* $\log X_1$.

We begin by writing down the power function. We have

$$\begin{aligned}\gamma(\beta) &= P_\beta(T < c)\\ &= P_\beta(\textstyle\prod_{i=1}^n X_i < c)\\ &= P_\beta(\textstyle\sum_{i=1}^n \log X_i < \log c)\\ &= P(W < \log c), \quad W \sim \text{Gamma}(n, 1/\beta),\end{aligned}$$

where we have used the facts that $\log X_1 \sim \text{Exponential}(1/\beta)$, which we can obtain using the univariate transformation method, and $\sum_{i=1}^n \log X_i \sim \text{Gamma}(n, 1/\beta)$, which we can obtain using moment generating functions.

In order to choose $c$ such that the rejection rule $T < c$ defines a size $\alpha$ test, we set

$$\alpha = \sup_{\beta\in\{\beta_0\}} \gamma(\beta) = \gamma(\beta_0) = P(W < \log c), \quad W \sim \text{Gamma}(n, 1/\beta_0).$$

The above shows that $\log c = G_{n,\theta_0^{-1},1-\alpha}$, where $G_{n,\theta_0^{-1},1-\alpha}$ is the upper $1-\alpha$ quantile of the Gamma$(n, 1/\theta_0)$ distribution. The size-$\alpha$ rejection rule is thus

$$\prod_{i=1}^{n} X_i < \exp(G_{n,\theta_0^{-1},1-\alpha}).$$

This is equivalent to the rule

$$\sum_{i=1}^{n} \log X_i < G_{n,\theta_0^{-1},1-\alpha}.$$

3. Let $X$ be a random variable with distribution determined by the hierarchical model

$$\begin{aligned} X|Z &\sim \text{Normal}(0, Z + (1-Z)\pi^2) \\ Z &\sim \text{Bernoulli}(\delta), \end{aligned}$$

for some $\delta \in [0,1]$. Consider testing $H_0$: $\delta = 1$ versus $H_1$: $\delta < 1$ with a single realization of $X$.

(a) Give an expression for the cdf $F_X(x;\delta) = P_\delta(X \leq x)$.
*Hint:* $\{X \leq x\} = \{X \leq x \cap Z = 0\} \cup \{X \leq x \cap Z = 1\}$.

> We have
>
> $$\begin{aligned} P_\delta(X \leq x) &= P_\delta(X \leq x \cap Z = 0) + P_\delta(X \leq x \cap Z = 1) \\ &= P_\delta(X \leq x | Z = 0)P(Z = 0) + P_\delta(X \leq x | Z = 1)P(Z = 1) \\ &= \Phi(x/\pi)(1-\delta) + \Phi(x)\delta, \end{aligned}$$
>
> where $\Phi$ is the cdf of the Normal$(0,1)$ distribution.

(b) Give the power function of the test which rejects $H_0$ when $|X| > c$ for some $c > 0$.

> We have
>
> $$\begin{aligned} \gamma(\delta) &= P_\delta(|X| > c) \\ &= P_\delta(X > c) + P(X < -c) \\ &= 1 - [\Phi(c/\pi)(1-\delta) + \Phi(c)\delta] + \Phi(-c/\pi)(1-\delta) + \Phi(-c)\delta \\ &= \delta \cdot 2[1 - \Phi(c)] + (1-\delta) \cdot 2[1 - \Phi(c/\pi)]. \end{aligned}$$

(c) Find the value of $c$ such that the test has size $\alpha$.

> We write
>
> $$\alpha = \sup_{\delta \in \{1\}} \gamma(\delta) = \gamma(1) = 2[1 - \Phi(c)],$$
>
> which gives $c = z_{\alpha/2}$, where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the Normal$(0,1)$ distribution.