STAT 713 sp 2022 Final Exam

1. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} f_Y(y;\theta) = \theta e^{-y\theta} e^{\theta} \mathbf{1}(y > 1)$ and consider testing H_0: $\theta = \theta_0$ vs H_1: $\theta \neq \theta_0$.
   (a) Find the MLE $\hat{\theta}_n$ for θ.
   (b) Show that the likelihood ratio test rejects H_0 when $\theta_0/\hat{\theta}_n < c_1$ or $\theta_0/\hat{\theta}_n > c_2$ for some $c_1 < c_2$.
   (c) Give the rejection rule for the size-α asymptotic likelihood ratio test.
   (d) Give the Fisher information.
   (e) Give the rejection rule for the size-α score test.

2. Let $X_1, \dots, X_n$ be iid rvs with the same distribution as X, of which the distribution is described by

$$\begin{aligned} X|\lambda &\sim \text{Poisson}(\lambda) \\ \lambda &\sim \text{Exponential}(\beta). \end{aligned}$$

for some $\beta > 0$.

(a) Show that the marginal pmf of X is $p_X(x;\beta) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta+1}\right)^x \mathbf{1}(x \in \{0, 1, 2, \dots\})$.

(b) Find the MLE $\hat{\beta}_n$ for β based on $X_1, \dots, X_n$.

(c) Find the asymptotic variance ϑ such that $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta)$ as $n \to \infty$.

(d) Give a Wald-type test of H_0: $\beta \leq \beta_0$ versus H_1: $\beta > \beta_0$.

(e) Give an asymptotic $(1 - \alpha) \times 100\%$ confidence interval for β.

3. Let $X_1, \ldots, X_n$ be independent rvs with cdf given by

$$F_X(x;\mu) = \begin{cases} 0, & x < \mu \\ 1 - e^{-(x-\mu)^2}, & x \geq \mu, \end{cases}$$

for some $\mu \in \mathbb{R}$.

(a) Find a sufficient statistic for μ.

(b) Find a pivotal quantity for μ.

(c) Give the cdf of your pivotal quantity.

(d) Use the pivot quantity to construct a $(1-\alpha) \times 100\%$ confidence interval for μ.