STAT 713 sp 2022 Final Exam

1. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} f_Y(y;\theta) = \theta e^{-y\theta} e^{\theta} \mathbf{1}(y > 1)$ and consider testing H_0: $\theta = \theta_0$ vs H_1: $\theta \neq \theta_0$.

   (a) Find the MLE $\hat{\theta}_n$ for θ.

   The likelihood and log-likelihood functions are given by

   $$\mathcal{L}(\theta; \mathbf{X}) = \theta^n e^{-n\bar{Y}_n\theta} e^{n\theta} \quad \text{and} \quad \ell(\theta; \mathbf{X}) = n \log \theta - n\theta(\bar{Y}_n - 1).$$

   The score function is given by

   $$S(\theta; \mathbf{X}) = \frac{\partial}{\partial \theta} \ell(\theta; \mathbf{X}) = n \left[\frac{1}{\theta} - (\bar{Y}_n - 1) \right]$$

   Setting this equal to zero and solving for θ gives the MLE

   $$\hat{\theta}_n = \frac{1}{\bar{Y}_n - 1}.$$

   (b) Show that the likelihood ratio test rejects H_0 when $\theta_0/\hat{\theta}_n < c_1$ or $\theta_0/\hat{\theta}_n > c_2$ for some $c_1 < c_2$.

   The likelihood ratio is given by

   $$\begin{aligned} \text{LR}(\mathbf{Y}) &= \frac{\mathcal{L}(\theta_0; \mathbf{X})}{\mathcal{L}(\hat{\theta}_n; \mathbf{X})} \\ &= \frac{\theta_0^n e^{-n\bar{Y}_n\theta_0} e^{n\theta_0}}{\hat{\theta}_n^n e^{-n\bar{Y}_n\hat{\theta}_n} e^{n\hat{\theta}_n}} \\ &= \left(\frac{\theta_0}{\hat{\theta}_n} \right)^n \exp\left[-n\theta_0(\bar{Y}_n - 1) + n\hat{\theta}_n(\bar{Y}_n - 1) \right] \\ &= \left[\left(\frac{\theta_0}{\hat{\theta}_n} \right) \exp\left[- \left(\frac{\theta_0}{\hat{\theta}_n} \right) \right] e^1 \right]^n. \end{aligned}$$

   So the likelihood ratio test rejects H_0: $\theta = \theta_0$ if

   $$\left[\left(\frac{\theta_0}{\hat{\theta}_n} \right) \exp\left[- \left(\frac{\theta_0}{\hat{\theta}_n} \right) \right] e^1 \right]^n < k$$

   for some $k > 0$. Since the function ze^{-z} is strictly increasing for $z < 1$ and strictly decreasing for $z > 1$, the rejection rule of the likelihood ratio test is equivalent to

   $$\frac{\theta_0}{\hat{\theta}_n} < c_1 \quad \text{or} \quad \frac{\theta_0}{\hat{\theta}_n} > c_2$$

for some $c_1 < c_2$.

(c) Give the rejection rule for the size-α asymptotic likelihood ratio test.

The size-α asymptotic likelihood ratio test rejects H_0: $\theta = \theta_0$ if

$$-2n\left[\log\left(\frac{\theta_0}{\hat{\theta}_n}\right) + \left(1 - \frac{\theta_0}{\hat{\theta}_n}\right)\right] > \chi^2_{1,\alpha}.$$

(d) Give the Fisher information.

The Fisher information is

$$I_n(\theta) = -\mathbb{E}\frac{\partial^2}{\partial\theta^2}\ell(\theta;\mathbf{X}) = \frac{n}{\theta^2}.$$

(e) Give the rejection rule for the size-α score test.

The score test statistic is given by

$$\frac{[S(\theta_0;\mathbf{X})]^2}{I_n(\theta_0)} = \frac{n^2[\theta_0^{-1} - (\bar{Y}_n - 1)]^2}{n\theta_0^{-2}} = n[1 - \theta_0(\bar{Y}_n - 1)]^2 = n[1 - \theta_0/\hat{\theta}_n]^2$$

The size-α score test rejects H_0: $\theta = \theta_0$ when

$$n[1 - \theta_0/\hat{\theta}_n]^2 > \chi^2_{1,\alpha}.$$

2. Let $X_1, \dots, X_n$ be iid rvs with the same distribution as X, of which the distribution is described by

$$X|\lambda \sim \text{Poisson}(\lambda)$$
$$\lambda \sim \text{Exponential}(\beta).$$

for some $\beta > 0$.

(a) Show that the marginal pmf of X is $p_X(x;\beta) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta+1}\right)^x \mathbf{1}(x \in \{0,1,2,\dots\})$.

For $x = 0, 1, 2, \dots,$ we have

$$\begin{aligned} p_X(x;\beta) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{1}{\beta}e^{-\lambda/\beta}d\lambda \\ &= \frac{\Gamma(x+1)}{x!\beta(1+1/\beta)^{x+1}}\int_0^\infty \frac{1}{(1+1/\beta)^{-(x+1)}\Gamma(x+1)}e^{-\lambda/(1+1/\beta)^{-1}}d\lambda \\ &= \frac{1}{\beta+1}\left(\frac{\beta}{\beta+1}\right)^x. \end{aligned}$$

(b) Find the MLE $\hat{\beta}_n$ for β based on $X_1, \dots, X_n$.

The likelihood function is

$$\mathcal{L}(\beta;\mathbf{X}) = \left(\frac{1}{\beta+1}\right)^n\left(\frac{\beta}{\beta+1}\right)^{n\bar{X}_n}$$

and the log-likelihood is

$$\ell(\beta;\mathbf{X}) = -n\log(\beta+1) + n\bar{X}_n[\log(\beta) - \log(\beta+1)].$$

Now we have

$$\frac{\partial}{\partial\beta}\ell(\beta;\mathbf{X}) = -\frac{n}{\beta+1} + n\bar{X}_n\left[\frac{1}{\beta} - \frac{1}{\beta+1}\right] = n\frac{\bar{X}_n - \beta}{\beta(\beta+1)}.$$

Setting the above equal to 0 and solving for β gives

$$\hat{\beta}_n = \bar{X}_n.$$

(c) Find the asymptotic variance ϑ such that $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\text{D}} \text{Normal}(0,\vartheta)$ as $n \to \infty$.

We first need to find the variance of X, which is mostly easily found as

$$\text{Var}\,X = \text{Var}(\mathbb{E}[X|\lambda]) + \mathbb{E}(\text{Var}[X|\lambda]) = \text{Var}\,\lambda + \mathbb{E}\lambda = \beta^2 + \beta = \beta(\beta+1).$$

The Score function is

$$S(\beta;\mathbf{X}) = n\frac{\bar{X}_n - \beta}{\beta(\beta+1)}$$

and Fisher information is

$$I_n(\beta) = \operatorname{Var} S(\beta; \mathbf{X}) = \frac{n^2}{\beta^2(\beta+1)^2}\frac{1}{n}\operatorname{Var} X = \frac{n}{\beta(\beta+1)}.$$

The asymptotic variance ϑ is equal to the inverse of the Fisher information based on a sample of size 1, so that

$$\vartheta = \beta(\beta+1).$$

(d) Give a Wald-type test of H_0: $\beta \le \beta_0$ versus H_1: $\beta > \beta_0$.

Wald type tests could be defined with rejection rule $Z > z_\alpha$, where

$$Z = \sqrt{n}(\hat{\beta} - \beta_0)/\sqrt{\hat{\beta}_n(\hat{\beta}_n + 1)} \quad \text{or} \quad Z = \sqrt{n}(\hat{\beta} - \beta_0)/\sqrt{\beta_0(\beta_0 + 1)}.$$

(e) Give an asymptotic $(1-\alpha)\times 100\%$ confidence interval for β.

The Wald-type $(1-\alpha)\times 100\%$ confidence interval is given by

$$\hat{\beta}_n \pm z_{\alpha/2}\sqrt{\hat{\beta}_n(\hat{\beta}_n + 1)/n}.$$

3. Let $X_1, \ldots, X_n$ be independent rvs with cdf given by

$$F_X(x;\mu) = \begin{cases} 0, & x < \mu \\ 1 - e^{-(x-\mu)^2}, & x \geq \mu, \end{cases}$$

for some $\mu \in \mathbb{R}$.

(a) Find a sufficient statistic for μ.

The pdf corresponding to the cdf F_X is given by

$$f_X(x;\mu) = 2(x-\mu)e^{-(x-\mu)^2}\mathbf{1}(x \geq \mu).$$

The joint density of $X_1, \ldots, X_n$ is given by

$$f_{\mathbf{X}}(\mathbf{x};\mu) = 2^n \cdot \prod_{i=1}^{n}(X_i - \mu) \cdot e^{-\sum_{i=1}^{n}(X_i-\mu)^2} \cdot \mathbf{1}(X_{(1)} > \mu).$$

The greatest reduction of the data we can achieve without losing information about μ is to keep the order statistics $T = (X_{(1)}, \ldots, X_{(n)})$.

(b) Find a pivotal quantity for μ.

We find that the first order statistic $X_{(1)}$ has density given by

$$f_{X_{(1)}}(x;\mu) = 2n(x-\mu)e^{-n(x-\mu)^2}\mathbf{1}(x > \mu),$$

in which μ is a location parameter. This suggests as a pivotal quantity

$$Q(\mathbf{X};\mu) = X_{(1)} - \mu,$$

which has density given by

$$f_Q(y;\mu) = 2nye^{-ny^2}\mathbf{1}(y > 0).$$

Note that this is free of the parameter μ, so it is indeed a pivotal quantity. It is possible to define other pivotal quantities.

(c) Give the cdf of your pivotal quantity.

The cdf of the pivotal quantity $Q(\mathbf{X};\mu) = X_{(1)} - \mu$ is given by

$$F_Q(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-ny^2} & y \geq 0, \end{cases}$$

(d) Use the pivot quantity to construct a $(1-\alpha) \times 100\%$ confidence interval for μ.

The u quantile q_u of the pivot quantity $Q(\mathbf{X};\mu) = X_{(1)} - \mu$ is given by

$$q_u = \sqrt{\frac{1}{n}\log\left(\frac{1}{1-u}\right)},$$

which we obtain by solving $F_Q(q_u) = u$ for q_u. So we may write

$$P\left(\sqrt{\frac{1}{n}\log\left(\frac{1}{1-\alpha/2}\right)} \leq X_{(1)} - \mu \leq \sqrt{\frac{1}{n}\log\left(\frac{1}{1-(1-\alpha/2)}\right)}\right) = 1-\alpha.$$

This is equavalent to

$$P\left(X_{(1)} - \sqrt{\frac{1}{n}\log\left(\frac{1}{\alpha/2}\right)} \leq \mu \leq \sqrt{\frac{1}{n}\log\left(\frac{1}{1-\alpha/2}\right)}\right) = 1-\alpha,$$

so that

$$\left(X_{(1)} - \sqrt{\frac{1}{n}\log\left(\frac{1}{\alpha/2}\right)}, X_{(1)} - \sqrt{\frac{1}{n}\log\left(\frac{1}{1-\alpha/2}\right)}\right)$$

is a $(1-\alpha)\times 100\%$ confidence interval for μ.