STAT 713 sp 2023 Exam 1

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f(x;\beta) = \beta^2 x e^{-\beta x}\mathbf{1}(x > 0)$ for some $\beta > 0$.
   (a) Find the maximum likelihood estimator for β.
   (b) Find the maximum likelihood estimator for $\tau = \tau(\beta) = 1/\beta$.
   (c) Check whether $T(X_1, \dots, X_n) = (\prod_{i=1}^n X_i, \sum_{i=1}^n X_i)$ is a minimal sufficient statistic.
   (d) Find the value of $\mathrm{Cov}(\hat{\beta}_n, S_n\hat{\beta}_n)$, where $\hat{\beta}_n$ is the MLE for β and S_n^2 is the sample variance.

2. A randomly selected spectator of a USC basketball game will shoot free throws until making two baskets. Suppose the ability $\theta \in (0,1)$ of a randomly selected spectator has the pdf $\pi(\theta) = 6\theta(1-\theta)\mathbf{1}(0 < \theta < 1)$ and, given the spectator's ability θ, the number of shots Y required by the spectator to make two baskets has pmf $p(y|\theta) = (y-1)\theta^2(1-\theta)^{y-2}\mathbf{1}(y \in \{2, 3, \dots\})$.

   (a) Give the posterior distribution of θ given Y.

   (b) Give a Bayesian estimate of the ability θ of a spectator who makes the 2nd basket on the 5th shot.

   (c) Give the value of the maximum likelihood estimator of θ for the same spectator (treat θ as fixed).

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} f(x;\rho) = (1-\sqrt{\rho})^{-1}\mathbf{1}(\sqrt{\rho} \le x \le 1)$.

(a) Find the method of moments estimator of ρ.

(b) Find the bias of the method of moments estimator.

(c) Find the maximum likelihood estimator of ρ.

(d) Which estimator uses all the information in the sample about the parameter? Justify your answer.