# STAT 713 sp 2023 Exam 2

1. Let $Y_1, \ldots, Y_n \overset{\text{ind}}{\sim} \text{Binomial}(2, p)$ and consider estimating $\tau(p) = (1-p)^2$.
   (a) Find a complete sufficient statistic for $p$.
   (b) Check whether the estimator $\tilde{\tau} = \mathbf{1}(Y_1 = 0)$ is unbiased for $\tau(p)$.
   (c) Find the estimator $\hat{\tau}(t) = \mathbb{E}[\tilde{\tau} | T = t]$, where $T$ is a complete sufficient statistic.
   (d) Give some properties of the estimator $\hat{\tau}$. Why are we interested in this estimator?
   *You can answer this question even if you get stuck on part* (c).

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x;\theta) = (\log\theta)\theta^{-x}\mathbf{1}(x > 0)$. Note that $\mathbb{E}X_1^k = k!(\log\theta)^{-k}$ for each $k = 1, 2, \dots$

(a) Find the maximum likelihood estimator $\hat{\theta}_n$ of $\theta$.

(b) Find $\vartheta$ such that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta)$.

(c) Give the Cramér-Rao lower bound $[\tau'(\theta)]^2/I_n(\theta)$ for unbiased estimators of $\tau(\theta) = \log(\log\theta)$.

(d) Consider testing $H_0$: $\theta \geq \theta_0$ versus $H_1$: $\theta < \theta_0$. Give a decision rule such that no other decision rule guaranteeing the same or smaller size can give greater power when $\theta < \theta_0$.

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$.

   (a) Give careful arguments proving that $\hat{\sigma}_n = \sqrt{n^{-1} \sum_{i=1}^{n} X_i^2}$ is a consistent estimator of $\sigma$.

   (b) Consider testing $H_0$: $\sigma^2 \leq \sigma_0^2$ versus $H_1$: $\sigma^2 > \sigma_0^2$ with the decision rule $\hat{\sigma}_n^2 > c$. Choose $c$ so that the test has size $\alpha$.