STAT 713 sp 2023 Exam 2

1. Let $Y_1, \dots, Y_n \overset{\text{ind}}{\sim} \text{Binomial}(2, p)$ and consider estimating $\tau(p) = (1-p)^2$.

   (a) Find a complete sufficient statistic for p.

   > Writing $p_Y(y) = \binom{2}{y}(1-p)^2 \exp(y \log(p/(1-p)))$, that is in exponential family form, we see that $\sum_{i=1}^n Y_i$ is a complete sufficient statistic.

   (b) Check whether the estimator $\tilde{\tau} = \mathbf{1}(Y_1 = 0)$ is unbiased for $\tau(p)$.

   > The pmf of the Binomial$(2, p)$ distribution is $p_Y(y) = \binom{2}{y}p^y(1-p)^{2-y}$ for $y = 0, 1, 2$. So we have $\mathbb{E}\tilde{\tau} = \mathbb{E}[\mathbf{1}(Y_1 = 0)] = P(Y_1 = 0) = (1-p)^2$, so $\tilde{\tau}$ is an unbiased estimator of $\tau(p) = (1-p)^2$.

   (c) Find the estimator $\hat{\tau}(t) = \mathbb{E}[\tilde{\tau}|T = t]$, where T is a complete sufficient statistic.

   > Letting $T = \sum_{i=1}^n Y_i$, we have
   >
   > $$\begin{aligned} \hat{\tau}(t) &= \mathbb{E}[\tilde{\tau}|T = t] \\ &= P(Y_1 = 0 | \textstyle\sum_{i=1}^n Y_i = t) \\ &= P(Y_1 = 0 \cap \textstyle\sum_{i=1}^n Y_i = t)/P(\textstyle\sum_{i=1}^n Y_i = t) \\ &= P(Y_1 = 0)P(\textstyle\sum_{i=2}^n Y_i = t)/P(\textstyle\sum_{i=1}^n Y_i = t) \\ &= \frac{(1-p)^2\binom{2(n-1)}{t}p^t(1-p)^{2(n-1)-t}}{\binom{2n}{t}p^t(1-p)^{2n-t}} \\ &= \frac{\binom{2(n-1)}{t}}{\binom{2n}{t}} \\ &= \frac{2(n-1)!/[(2(n-1)-t)!t!])}{2n!/[(2n-t)!t!]} \\ &= \frac{(2n-t)(2n-t-1)}{2n(2n-1)}, \end{aligned}$$
   >
   > where we have used $\sum_{i=1}^n Y_i \sim \text{Binomial}(2n, p)$ and $\sum_{i=2}^n Y_i \sim \text{Binomial}(2(n-1), p)$. So we have
   >
   > $$\hat{\tau} = \frac{1}{2n}(2n - \textstyle\sum_{i=1}^n Y_i)(2n - \textstyle\sum_{i=1}^n Y_i - 1).$$

   (d) Give some properties of the estimator $\hat{\tau}$. Why are we interested in this estimator?
   You can answer this question even if you get stuck on part (c).

We know that the estimator $\hat{\tau}$ is unbiased; moreover it has the smallest variance of any unbiased estimator for τ. The Rao-Blackwell and Lehmann-Scheffé results give us this.

2. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x;\theta) = (\log\theta)\theta^{-x}\mathbf{1}(x > 0)$. Note that $\mathbb{E}X_1^k = k!(\log\theta)^{-k}$ for each $k = 1, 2, \dots$

(a) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ.

The likelihood function is given by

$$\mathcal{L}(\theta; \mathbf{X}) = (\log\theta)^n \theta^{-n\bar{X}_n}$$

and the log-likelihood is given by

$$\ell(\theta; \mathbf{X}) = n\log(\log\theta) - n\bar{X}_n \log\theta.$$

The score function is

$$S(\theta; \mathbf{X}) = \frac{\partial}{\partial\theta}\ell(\theta; \mathbf{X}) = \frac{n}{\theta\log\theta} - \frac{n\bar{X}_n}{\theta}.$$

The value of θ which sets the score function equal to zero is $\hat{\theta}_n = e^{1/\bar{X}_n}$.

(b) Find ϑ such that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta)$.

The asymptotic variance ϑ is equal to $1/I_1(\theta)$, where $I_1(\theta)$ is the Fisher information based on a sample of size 1. We have

$$I_1(\theta) = \operatorname{Var} S(\theta; X_1) = \operatorname{Var}(X_1/\theta) = \frac{1}{\theta^2}\operatorname{Var} X_1 = \frac{1}{\theta^2}\left[\frac{2}{(\log\theta)^2} - \left(\frac{1}{\log\theta}\right)^2\right] = \frac{1}{\theta^2(\log\theta)^2}.$$

Therefore

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{D}} \text{Normal}(0, \theta^2(\log\theta)^2)$$

as $n \to \infty$.

(c) Give the Cramér-Rao lower bound $[\tau'(\theta)]^2/I_n(\theta)$ for unbiased estimators of $\tau(\theta) = \log(\log\theta)$.

We have $I_n(\theta) = n/[\theta^2(\log\theta)^2]$ and $\tau'(\theta) = 1/[(\log\theta)\theta]$, so the CRLB for unbiased estimators of $\tau(\theta)$ is given by

$$\frac{[\tau'(\theta)]^2}{I_n(\theta)} = \frac{(1/[(\log\theta)\theta])^2}{n/[\theta^2(\log\theta)^2]} = \frac{1}{n}.$$

(d) Consider testing H_0: $\theta \geq \theta_0$ versus H_1: $\theta < \theta_0$. Give a decision rule such that no other decision rule guaranteeing the same or smaller size can give greater power when $\theta < \theta_0$.

For any $\theta_1 < \theta_0$ the ratio

$$\frac{\mathcal{L}(\theta_0; \mathbf{X})}{\mathcal{L}(\theta_1; \mathbf{X})} = \frac{(\log\theta_0)^n \theta_0^{-n\bar{X}_n}}{(\log\theta_1)^n \theta_1^{-n\bar{X}_n}} = \left(\frac{\log\theta_0}{\log\theta_1}\right)^n \left(\frac{\theta_1}{\theta_0}\right)^{n\bar{X}_n}$$

is monotone decreasing in $\bar{X}_n$, which is a sufficient statistic for θ. Therefore the UMP test rejects when $\bar{X}_n > c$ for some c.

3. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$.

(a) Give careful arguments proving that $\hat{\sigma}_n = \sqrt{n^{-1} \sum_{i=1}^n X_i^2}$ is a consistent estimator of σ.

> The weak law of large numbers gives $n^{-1} \sum_{i=1}^n X_i \xrightarrow{p} \sigma^2$, since $\mathbb{E}X_i^2 = \sigma^2$. Since the function $g(z) = \sqrt{z}$ is a continuous function, we have $g(n^{-1} \sum_{i=1}^n X_i^2) \xrightarrow{p} g(\sigma^2)$; that is $\sqrt{n^{-1} \sum_{i=1}^n X_i^2} \xrightarrow{p} \sigma$.

(b) Consider testing H_0: $\sigma^2 \leq \sigma_0^2$ versus H_1: $\sigma^2 > \sigma_0^2$ with the decision rule $\hat{\sigma}_n^2 > c$. Choose c so that the test has size α.

> The power function of the test is given by
>
> $$\begin{aligned} \gamma(\sigma^2) &= P_{\sigma^2}(n^{-1} \textstyle\sum_{i=1}^n X_i^2 > c) \\ &= P_{\sigma^2}(\textstyle\sum_{i=1}^n (X_i/\sigma)^2 > n(c/\sigma^2)) \\ &= P(W_n > n(c/\sigma^2)), \quad \text{where } W_n \sim \chi_n^2. \end{aligned}$$
>
> Setting the size of the test equal to α, we obtain
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> $$\alpha = \sup_{\sigma^2 \leq \sigma_0^2} = \gamma(\sigma_0^2) = P(W_n > n(c/\sigma_0^2)) \iff \frac{nc}{\sigma_0^2} = \chi_{n,\alpha}^2,$$
>
> where $\chi_{n,\alpha}^2$ is the upper $\alpha/2$-quantile of the χ_n^2 distribution. So setting
>
> $$c = \frac{\sigma_0^2}{n} \chi_{n,\alpha}^2$$
>
> calibrates the test to have size α.