STAT 713 sp 2023 Final Exam

1. Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} f_X(x; \beta) = 2x\beta^{-2}e^{-x^2/\beta^2}\mathbf{1}(x > 0)$ for some $\beta > 0$.
   (a) Find the MLE $\hat{\beta}_n$ for β.
   (b) Find $\mathbb{E}X_1^2$ using the fact that the score function has expectation 0.
   (c) Give the Fisher information $I_n(\beta)$.
   (d) Give the value ϑ such that $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\text{D}} \text{Normal}(0, \vartheta)$ as $n \to \infty$.
   (e) Give a Wald-type $(1 - \alpha) \times 100\%$ confidence interval for β.
   (f) Give the size-α asymptotic likelihood ratio test for H_0: $\beta = \beta_0$ versus H_1 $\beta \neq \beta_0$.

2. Let $Y_1, \ldots, Y_n \overset{\text{ind}}{\sim} \text{Bernoulli}(e^\theta/(1+e^\theta))$ for some $\theta \in \mathbb{R}$.

   (a) Give the MLE $\hat{\theta}_n$ for θ. *Hint: Use the invariance property of MLEs.*

   (b) Find the Fisher information $I_n(\theta)$.

   (c) Give the size-α score test for testing H_0: $\theta = 0$ versus H_1: $\theta \neq 0$.

   (d) Give an expression for the set of θ_0 values such that the size-α score test would fail to reject H_0: $\theta = \theta_0$ in favor of H_1: $\theta \neq \theta_0$. What is the purpose for collecting these values of θ_0, i.e. what can this set be used for?

3. Let $X_1, \ldots, X_n \overset{\text{ind}}{\sim}$ Exponential(β) with β unknown. Consider testing H_0: $\beta \le \beta_0$ versus H_1: $\beta > \beta_0$ with the rejection rule $\sqrt{n}(\bar{X}_n/\beta_0 - 1) > c$ for some c.

   (a) Find the value of c under which the test will have size exactly α.

   (b) Give the value of c under which the test will have size approaching α as $n \to \infty$.

   (c) Give the asymptotic one-sided confidence interval for β based on inverting the asymptotic size-α one-sided test in part (b).

   (d) Find an approximation to the smallest sample size needed to reject H_0: $\beta \le \beta_0$ with probability at least γ^* when $\beta = \beta^*$ for some $\beta^* > \beta_0$. *Your answer should involve some Normal quantiles.*