1

$X_1, \dots, X_n \overset{ind}{\sim} f_X(x;\beta) = \frac{2x}{\beta^2} e^{-x^2/\beta^2} \mathbb{1}(x>0).$

(a)

$$L(\beta; \underset{\sim}{X}) = \left(\frac{2}{\beta^2}\right)^n \left(\prod_{i=1}^n X_i\right) e^{-\frac{1}{\beta^2}\sum_{i=1}^n X_i^2}$$

$$\ell(\beta;\underset{\sim}{X}) = n\log 2 - 2n\log\beta + \sum_{i=1}^n \log X_i - \frac{1}{\beta^2}\sum_{i=1}^n X_i^2.$$

$$S(\beta;\underset{\sim}{X}) = \frac{\partial}{\partial\beta}\ell(\beta;\underset{\sim}{X}) = -\frac{2n}{\beta} + \frac{2}{\beta^3}\sum_{i=1}^n X_i^2 \overset{set}{=} 0$$

$$\Rightarrow \quad \beta^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 \quad \text{so} \quad \boxed{\hat{\beta}_n = \sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}}$$

(b)

$$\mathbb{E}\, S(\beta;\underset{\sim}{X}) = 0 \iff -\frac{2n}{\beta} + \frac{2}{\beta^3}\sum_{i=1}^n \mathbb{E}X_i^2 = 0$$

$$\iff -\frac{2n}{\beta} + \frac{2n\,\mathbb{E}X_1^2}{\beta^3} = 0$$

$$\iff \beta^2 = \mathbb{E}X_1^2.$$

(c) $$I_n(\beta) = -\mathbb{E}\,\frac{\partial}{\partial\theta} S(\theta ; \underset{\sim}{X})$$

$$= -\mathbb{E}\left[\frac{2n}{\beta^2} - \frac{2(3)}{\beta^4}\sum_{i=1}^{n} X_i^2\right]$$

$$= \frac{-2n}{\beta^2} + \frac{6}{\beta^4}\, n\, \beta^2$$

$$= \frac{4n}{\beta^2}.$$

(d) $\vartheta = \frac{1}{I_1(\beta)} = \frac{\beta^2}{4}$. That is, $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{D} \text{Normal}\left(0, \frac{\beta^2}{4}\right)$.

(e) Using $\hat{\vartheta} = \frac{\hat{\beta}_n^2}{4}$, an asymptotic $(1-\alpha)^* 100\%$ C.I. is

$$\hat{\beta}_n \pm z_{\alpha/2}\sqrt{\frac{\hat{\beta}_n^2}{4n}}, \quad \text{where } \hat{\beta}_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}.$$

(f) The likelihood ratio is

$$LR(\underset{\sim}{X}) = \frac{\left(\frac{2}{\beta_0^2}\right)^n \left(\prod_{i=1}^n x_i\right) e^{-\frac{1}{\beta_0^2}\sum_{i=1}^n x_i^2}}{\left(\frac{2}{\hat\beta_n^2}\right)^n \left(\prod_{i=1}^n x_i\right) e^{-\frac{1}{\hat\beta_n^2}\sum_{i=1}^n x_i^2}}$$

$$= \left(\frac{\hat\beta_n}{\beta_0}\right)^{2n} \left[e^{-\left(\frac{\hat\beta_n}{\beta_0}\right)^2}\right]^n e^n$$

$$= \left[\left(\frac{\hat\beta_n^2}{\beta_0^2}\right) \exp\left(-\frac{\hat\beta_n^2}{\beta_0^2}\right)\right]^n e^n .$$

For the ALRT we compute

$$-2\log LR(\underset{\sim}{X}) = -2n \log\left(\frac{\hat\beta_n^2}{\beta_0^2}\right) + 2n\frac{\hat\beta_n^2}{\beta_0^2} - 2n$$

$$= 2n\left[\left(\frac{\hat\beta_n^2}{\beta_0^2} - 1\right) - \log\left(\frac{\hat\beta_n^2}{\beta_0^2}\right)\right].$$

The ALRT rejects $H_0: \beta = \beta_0$ when

$$2n\left[\left(\frac{\hat{\beta}_n^2}{\beta_0^2} - 1\right) - \log\left(\frac{\hat{\beta}_n^2}{\beta_0^2}\right)\right] \geq \chi^2_{1,\alpha}.$$

$\boxed{2}$ $Y_1, \dots, Y_n \overset{ind}{\sim} \text{Bernoulli}\left(\frac{e^\theta}{1+e^\theta}\right)$.

(a) Let $p = \frac{e^\theta}{1+e^\theta}$. Then the MLE for p is $\bar{Y}_n$. i.e. $\hat{p}_n = \bar{Y}_n$.

We have $\theta = \log\left(\frac{p}{1-p}\right)$, so $\boxed{\hat{\theta}_n = \log\left(\frac{\hat{p}_n}{1-\hat{p}_n}\right) = \log\left(\frac{\bar{Y}_n}{1-\bar{Y}_n}\right).}$

~~(b)~~ $$L(\theta; \underset{\sim}{Y}) = \prod_{i=1}^n \left(\frac{e^\theta}{1+e^\theta}\right)^{Y_i}\left(1 - \frac{e^\theta}{1+e^\theta}\right)^{1-Y_i}$$

$$= \left(\frac{e^\theta}{1+e^\theta}\right)^{n\bar{Y}_n}\left(1 - \frac{e^\theta}{1+e^\theta}\right)^{n-n\bar{Y}_n}.$$

For testing $H_0: \theta \leq 0$ vs $\theta > 0$, we have

$$LR(\underset{\sim}{Y}) = \frac{\sup_{\theta \leq 0} L(\theta; \underset{\sim}{Y})}{\sup_{\theta \in \mathbb{R}} L(\theta; \underset{\sim}{Y})}$$

$$= \begin{cases} 1 & \text{if } \hat{\theta}_n \leq 0 \\ \dfrac{L(0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} & \text{if } \hat{\theta}_n > 0, \end{cases}$$

where

$$\frac{L(0; \underset{\sim}{Y})}{L(\hat{\theta}_n; \underset{\sim}{Y})} = \frac{\left(\frac{e^0}{1+e^0}\right)^{n\bar{Y}_n} \left(1 - \frac{e^0}{1+e^0}\right)^{n - n\bar{Y}_n}}{\left(\frac{e^{\hat{\theta}_n}}{1+e^{\hat{\theta}_n}}\right)^{n\bar{Y}_n} \left(1 - \frac{e^{\hat{\theta}_n}}{1+e^{\hat{\theta}_n}}\right)^{n - n\bar{Y}_n}} = \frac{\left(\frac{1}{2}\right)^{n\bar{Y}_n} \left(1 - \frac{1}{2}\right)^{n - n\bar{Y}_n}}{\bar{Y}_n^{\,n\bar{Y}_n} \left(1 - \bar{Y}_n\right)^{n - n\bar{Y}_n}}$$

$$= \left(\frac{1/2}{\bar{Y}_n}\right)^{n\bar{Y}_n}\left(\frac{1-1/2}{1-\bar{Y}_n}\right)^{n-n\bar{Y}_n}.$$

So we reject H_0 if

$$\left(\frac{1/2}{\bar{Y}_n}\right)^{n\bar{Y}_n}\left(\frac{1-1/2}{1-\bar{Y}_n}\right)^{n-n\bar{Y}_n} < k,$$

for some $k \in [0,1]$, provided $\hat{\theta}_n > 0 \iff \bar{Y}_n > 1/2$.

~~(a)~~ (b) $$\ell(\theta;\underset{\sim}{X}) = n\bar{Y}_n \log\left(\frac{e^\theta}{1+e^\theta}\right) + (n-n\bar{Y}_n)\log\left(\underbrace{1-\frac{e^\theta}{1+e^\theta}}_{=\frac{1}{1+e^\theta}}\right)$$

$$= n\bar{Y}_n \log\left(e^\theta\right) + n\log\left(\frac{1}{1+e^\theta}\right)$$

$$= n\bar{Y}_n\theta - n\log\left(1+e^\theta\right).$$

$$S(\theta;\underline{X}) = n\bar{Y}_n - n\frac{e^\theta}{1+e^\theta}$$

$$= n\left(\bar{Y}_n - \frac{e^\theta}{1+e^\theta}\right).$$

$$I_n(\theta) = \mathrm{Var}\, S(\theta;\underline{X}) = n^2\,\mathrm{Var}\,\bar{Y}_n$$

$$= n^2\,\frac{\mathrm{Var}\,Y_1}{n}$$

$$= n\left(\frac{e^\theta}{1+e^\theta}\right)\underbrace{\left(1-\frac{e^\theta}{1+e^\theta}\right)}_{=\frac{1}{1+e^\theta}}$$

$$= n\,\frac{e^\theta}{(1+e^\theta)^2}.$$

~~(d)~~ (c) Score test:

$$\frac{[S(0;\underset{\sim}{Y})]^2}{I_n(0)} = \frac{[n(\bar{Y}_n - \frac{1}{2})]^2}{n\ \frac{1}{2}\cdot\frac{1}{2}}$$

$$= \frac{n\left(\bar{Y}_n - \frac{1}{2}\right)^2}{\frac{1}{2}\cdot\frac{1}{2}}.$$

Reject H_0 if $\dfrac{n\left(\bar{Y}_n - \frac{1}{2}\right)^2}{\frac{1}{2}\cdot\frac{1}{2}} > \chi^2_{1,\alpha}$.

~~(e)~~ (d) A $(1-\alpha)^* 100\%$ C.I. for θ is

$$\left\{ \theta_0 : \frac{[S(\theta_0;\underset{\sim}{Y})]^2}{I_n(\theta_0)} \le \chi^2_{1,\alpha} \right\}$$

$$= \left\{ \theta_0 : \frac{n\left(\bar{Y}_n - \frac{e^{\theta_0}}{1+e^{\theta_0}}\right)^2}{\frac{e^{\theta_0}}{1+e^{\theta_0}}\left(1 - \frac{e^{\theta_0}}{1+e^{\theta_0}}\right)} \le \chi^2_{1,\alpha} \right\}.$$

$\boxed{3}$ $X_1, \dots, X_n \overset{ind}{\sim} \text{Exponential}(\beta_0)$. $H_0: \beta \le \beta_0$ vs $H_1: \beta > \beta_0$.

(a) Power of test with rule $\sqrt{n}\left(\frac{\bar{X}_n}{\beta_0} - 1\right) > c$ is

$$\gamma_n(\beta) = P_\beta\left(\sqrt{n}\left(\frac{\bar{X}_n}{\beta_0} - 1\right) > c\right)$$

$$= P_\beta\left(\frac{\bar{X}_n}{\beta_0} > \frac{c}{\sqrt{n}} + 1\right)$$

$$\overset{\text{or}}{=} P_\beta\left(\frac{\bar{X}_n}{\beta} > \frac{\beta_0}{\beta}\left(\frac{c}{\sqrt{n}} + 1\right)\right)$$

$$= P_\beta\left(W > \frac{\beta_0}{\beta}\left(\frac{c}{\sqrt{n}} + 1\right)\right),$$

$W \sim \text{Gamma}\left(n, \frac{1}{n}\right)$.

$$= P_\beta\left(\bar{X}_n > \frac{c}{\sqrt{n}}\beta_0 + \beta_0\right), \quad \bar{X}_n \sim \text{Gamma}\left(n, \frac{\beta}{n}\right).$$

$$= 1 - P_\beta\left(\bar{X}_n \le \frac{c}{\sqrt{n}}\beta_0 + \beta_0\right).$$

$$= 1 - F_{\text{Gamma}\left(n, \frac{\beta}{n}\right)}\left(\frac{c}{\sqrt{n}}\beta_0 + \beta_0\right)$$

or $1 - F_{\text{Gamma}\left(n, \frac{1}{n}\right)}\left(\frac{\beta_0}{\beta}\left(\frac{c}{\sqrt{n}} + 1\right)\right)$

$F_{\text{Gamma}\left(n, \frac{\beta}{n}\right)}(\cdot)$ is the cdf of the $\text{Gamma}\left(n, \frac{\beta}{n}\right)$ distribution.

The size is given by

$$\sup_{\beta \le \beta_0} \gamma_n(\beta) = \gamma_n(\beta_0) = 1 - F_{\text{Gamma}\left(n, \frac{\beta_0}{n}\right)}\left(\frac{c}{\sqrt{n}}\beta_0 + \beta_0\right).$$

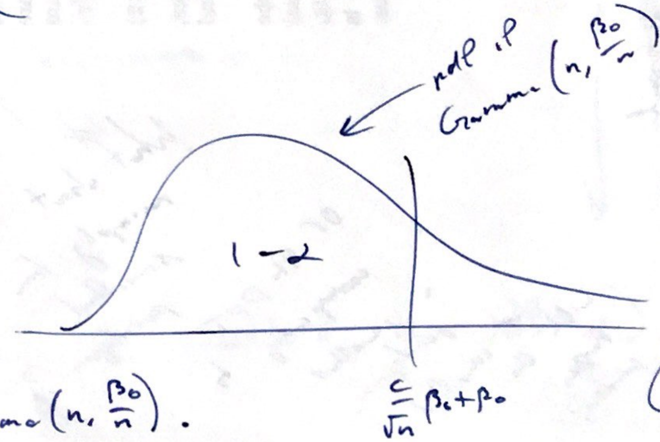
$$= 1 - F_{\text{Gamma}\left(n, \frac{1}{n}\right)}\left(\frac{c}{\sqrt{n}} + 1\right) = \alpha \iff \frac{c}{\sqrt{n}} + 1 = G_{n, \frac{1}{n}, \alpha}$$

$$\iff c = \sqrt{n}\left(G_{n, \frac{1}{n}, \alpha} - 1\right)$$

Set $\alpha = 1 - F_{\text{Gamma}\left(n, \frac{\beta_0}{n}\right)}\left(\frac{c}{\sqrt{n}}\beta_0 + \beta_0\right)$ and solve for c

$$\iff F_{\text{Gamma}\left(n, \frac{\beta_0}{n}\right)}\left(\frac{c}{\sqrt{n}}\beta_0 + \beta_0\right) = 1 - \alpha$$

$$\iff \frac{c}{\sqrt{n}}\beta_0 + \beta_0 = \underbrace{G_{n, \frac{\beta_0}{n}, \alpha}}_{\text{upper } \alpha \text{ quantile of Gamma}\left(n, \frac{\beta_0}{n}\right)}.$$



$\Leftrightarrow \quad c_\alpha = \sqrt{n}\left(\dfrac{G_{2n, \frac{\beta_0}{n}, \alpha} - \beta_0}{\beta_0}\right).$

$\boxed{c_\alpha = \sqrt{n}\left(G_{n, \frac{1}{n}, \alpha} - 1\right)}$

With this value of c the test will have size exactly $1-\alpha$.

(b) We have $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{D} \text{Normal}(0, \beta^2)$, where $\hat{\beta}_n = \bar{X}_n$.

Therefore

$$\gamma_n(\beta_0) = P_{\beta_0}\left(\sqrt{n}\left(\frac{\bar{X}_n}{\beta_0} - 1\right) > c\right)$$

$$= P_{\beta_0}\left(\frac{\sqrt{n}(\bar{X}_n - \beta_0)}{\beta_0} > c\right)$$

$$\longrightarrow P(Z > c) \quad \text{as } n \to \infty,$$

where $Z \sim N(0,1)$.

So choose $c_\alpha = z_\alpha$, the upper α quantile of the $N(0,1)$.

(c) An asymptotic $(1-\alpha)*100\%$ C.I. for β is

$$\left\{ \beta_0 : \sqrt{n}\left(\frac{\bar{X}_n}{\beta_0} - 1\right) \le Z_\alpha \right\}$$

$$= \left\{ \beta_0 : \frac{\sqrt{n}(\bar{X}_n - \beta_0)}{\beta_0} \le Z_\alpha \right\}$$

$$= \left\{ \beta_0 : \sqrt{n}\,\bar{X}_n - \sqrt{n}\beta_0 \le Z_\alpha \beta_0 \right\}$$

$$= \left\{ \beta_0 : \sqrt{n}\,\bar{X}_n \le (Z_\alpha + \sqrt{n})\beta_0 \right\}$$

$$= \left\{ \beta_0 : \frac{\sqrt{n}\,\bar{X}_n}{Z_\alpha + \sqrt{n}} \le \beta_0 \right\}$$

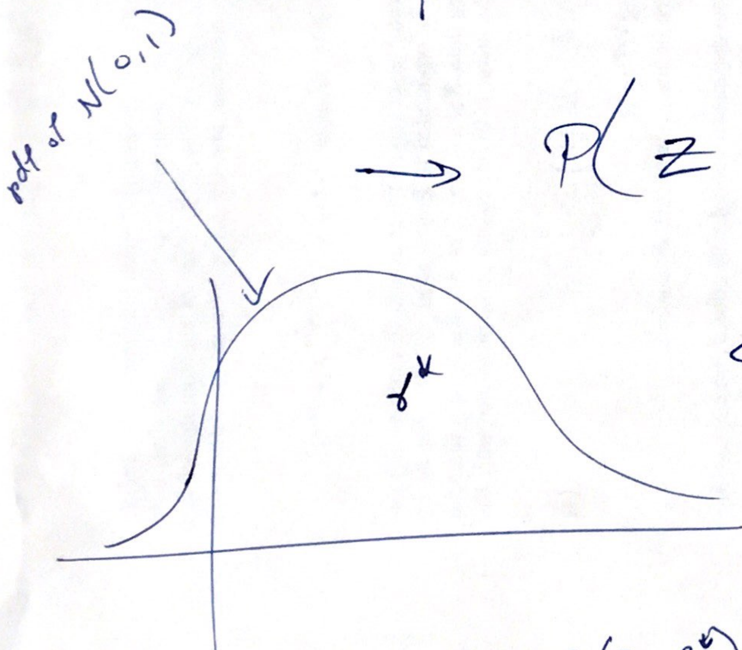
$$= \left[\left(\frac{\sqrt{n}}{\sqrt{n} + Z_\alpha}\right)\bar{X}_n, \ \infty\right).$$

(d) $\gamma_n(\beta^*) = P_{\beta^*}\left(\sqrt{n}\,\dfrac{(\bar{X}_n - \beta_0)}{\beta_0} > Z_\alpha \right)$

$$= P_{\beta^*}\left(\sqrt{n}\,\bar{X}_n > \beta_0 Z_\alpha + \sqrt{n}\,\beta_0 \right)$$

$$= P_{\beta^*}\left(\frac{\sqrt{n}\,(\bar{X}_n - \beta^*)}{\beta^*} > \frac{\beta_0 Z_\alpha}{\beta^*} + \frac{\sqrt{n}\,(\beta_0 - \beta^*)}{\beta^*} \right)$$

$$\longrightarrow P\left(Z > \frac{\beta_0 Z_\alpha}{\beta^*} + \frac{\sqrt{n}\,(\beta_0 - \beta^*)}{\beta^*} \right) \overset{\text{set}}{\geq} \gamma^*$$



$$\Leftrightarrow \quad Z_{\gamma^*} \geq \frac{\beta_0}{\beta^*} Z_\alpha - \frac{\sqrt{n}\,(\beta^* - \beta_0)}{\beta^*}$$

$$\Leftrightarrow \quad \beta^* Z_{\gamma^*} - \beta_0 Z_\alpha \geq -\sqrt{n}\,(\beta^* - \beta_0)$$

$$Z_{\gamma^*} \geq \frac{\beta_0 Z_\alpha}{\beta^*} + \frac{\sqrt{n}\,(\beta_0 - \beta^*)}{\beta^*}$$

$$\frac{\beta_0 Z_\alpha - \beta^* Z_{\gamma^*}}{\beta^* - \beta_0} \leq \sqrt{n}$$

$$\boxed{\left(\frac{\beta^* Z_{1-\gamma^*} + \beta_0 Z_\alpha}{\beta^* - \beta_0} \right)^2 \leq n.}$$