$\underset{\sim}{y} = X\underset{\sim}{b} + \underset{\sim}{e}$, $\underset{\sim}{e} \sim \text{Normal}(\underset{\sim}{0}, \sigma^2 I_n)$

$\text{or } V.$

# STAT 714 fa 2023 Lec 04

## Distributions of quadratic forms, Cochran's theorem, ANOVA table

$\underset{\sim}{y}^T A \underset{\sim}{y}$


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$\underset{\sim}{y} \sim \text{Normal}(\mu, V)$

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

## Univariate Normal distribution:

- The pdf of the Normal($\mu, \sigma^2$) distribution is given by

$$f(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[ -\frac{(y-\mu)^2}{2\sigma^2} \right] \quad \text{for } y \in \mathbb{R}.$$

- The moment generating function is $M_Y(t) = e^{t\mu + \sigma^2 t^2/2}$ for all $t \in \mathbb{R}$.
- The pdf and cdf of the Normal(0, 1) distribution get special notation:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

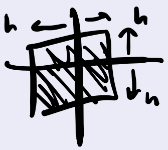
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad \text{for } z \in \mathbb{R}.$$

- The mgf of $Z \sim$ Normal(0, 1) is $M_Z(t) = e^{t^2/2}$ for all $t \in \mathbb{R}$.

For univariate $Y$, $M_Y(t) = \mathbb{E} e^{tY}$, provided that the expectation exists for $t$ in a neighborhood of $0$.

## Moment generating function of a random vector

The *moment generating function* of a $p \times 1$ random vector $\mathbf{y}$ is defined as



$$M_{\mathbf{y}}(\mathbf{t}) = \mathbb{E} e^{\mathbf{t}^T \mathbf{y}},$$

provided the expectation exists for all $\mathbf{t}$ such that $\|\mathbf{t}\|_\infty < h$ for some $h > 0$.

$\mathbb{E} e^{Z_j t_j} = e^{t^2/2}$ $\quad \|\mathbf{t}\|_\infty = \max_{1 \le j \le p} |t_j|$

**Exercise:** Let $Z_1, \dots, Z_p \overset{\text{ind}}{\sim} \text{Normal}(0,1)$ and set $\underset{p\times 1}{\mathbf{z}} = [Z_1, \dots, Z_p]^T$. Find the mgf of $\boxed{\mathbf{y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{z}.}$

$$M_{\mathbf{z}}(\mathbf{t}) = \mathbb{E} e^{\mathbf{t}^T \mathbf{z}} = \mathbb{E} e^{\sum_{j=1}^p t_j Z_j} = \mathbb{E} \prod_{j=1}^p e^{t_j Z_j} = \underset{\text{(independence of } Z_1, \dots, Z_p)}{\prod_{j=1}^p \mathbb{E} e^{t_j Z_j}} = \prod_{j=1}^p M_{Z_j}(t_j)$$

$$= \prod_{j=1}^p e^{t_j^2/2} = e^{\sum_{j=1}^p \frac{t_j^2}{2}} = \boxed{e^{\mathbf{t}^T \mathbf{t}/2}.}$$

$\underset{\sim}{y} = \underset{\sim}{\mu} + A\underset{\sim}{z}.$

$$M_{\underset{\sim}{y}}(\underset{\sim}{t}) = \mathbb{E}\, e^{\underset{\sim}{t}^T \underset{\sim}{y}}$$

$$= \mathbb{E}\, e^{\underset{\sim}{t}^T(\underset{\sim}{\mu} + A\underset{\sim}{z})}$$

$$= e^{\underset{\sim}{t}^T\underset{\sim}{\mu}}\ \mathbb{E}\, e^{\underset{\sim}{t}^T A \underset{\sim}{z}}$$

$$= e^{\underset{\sim}{t}^T\underset{\sim}{\mu}}\ M_{\underset{\sim}{z}}(A^T\underset{\sim}{t})$$

$$= e^{\underset{\sim}{t}^T\underset{\sim}{\mu}}\ e^{(A^T\underset{\sim}{t})^T A^T\underset{\sim}{t}/2}$$

$$\boxed{= e^{\underset{\sim}{t}^T\underset{\sim}{\mu} + \underset{\sim}{t}^T A A^T \underset{\sim}{t}/2}.}$$

is the mgf of the

$$\text{Normal}(\underset{\sim}{\mu},\ AA^T).$$

$$M_{\underset{\sim}{y}}(\underset{\sim}{t}) = \mathbb{E}\, e^{\underset{\sim}{t}^T \underset{\sim}{y}} = \mathbb{E}\, e^{\sum_{j=1}^{k} \underset{\sim}{t}_j^T \underset{\sim}{y}_j} = \mathbb{E} \prod_{j=1}^{k} e^{\underset{\sim}{t}_j^T \underset{\sim}{y}_j} = \prod_{j=1}^{k} \mathbb{E}\, e^{\underset{\sim}{t}_j^T \underset{\sim}{y}_j}$$

$$= \prod_{j=1}^{k} M_{\underset{\sim}{y}_j}(\underset{\sim}{t}_j)$$

## Result (Multivariate mgf results)

*Let* $\mathbf{y}_1, \ldots, \mathbf{y}_K$ *be* $K$ *random vectors with mgfs* $M_{\mathbf{y}_1}, \ldots, M_{\mathbf{y}_K}$*, respectively. Then*

1. $\mathbf{y}_1 \stackrel{d}{=} \mathbf{y}_2$ *iff* $M_{\mathbf{y}_1}(\mathbf{t}) = M_{\mathbf{y}_2}(\mathbf{t})$ *for all* $\|\mathbf{t}\|_\infty < h$ *for some* $h > 0$.
2. $\mathbf{y}_1, \ldots, \mathbf{y}_K$ *are mutually independent iff*

$$M_{\mathbf{y}}(\mathbf{t}) = M_{\mathbf{y}_1}(\mathbf{t}_1) \times \cdots \times M_{\mathbf{y}_K}(\mathbf{t}_K)$$

*for all* $\mathbf{t}$ *such that* $\|\mathbf{t}\|_\infty < h$ *for some* $h > 0$*, where* $\mathbf{y} = [\mathbf{y}_1^T \ldots \mathbf{y}_K^T]^T$ *and* $\mathbf{t} = [\mathbf{t}_1^T \ldots \mathbf{t}_K^T]^T$.

See Res 5.1 and 5.2 of Monahan (2008).

## Multivariate Normal distribution

$\underset{\sim}{y} \sim \text{Normal}(\underset{\sim}{\mu}, \Sigma)$

- The pdf of the Normal$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution is given by

$$f(\mathbf{y}; \boldsymbol{\mu}, \Sigma) = (2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right] \quad \text{for } \mathbf{y} \in \mathbb{R}^p.$$

- The moment generating function is $M_{\mathbf{y}}(\mathbf{t}) = e^{\mathbf{t}^T\boldsymbol{\mu}+\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}/2}$ for all $\mathbf{t} \in \mathbb{R}^p$.
- The pdf of the Normal$(\mathbf{0}, \mathbf{I}_p)$ distribution is

$$\phi(\mathbf{z}) = (2\pi)^{-p/2}e^{-\mathbf{z}^T\mathbf{z}/2} \quad \text{for } \mathbf{z} \in \mathbb{R}^p.$$

- The mgf of $\mathbf{z} \sim$ Normal$(\mathbf{0}, \mathbf{I}_p)$ is $M_{\mathbf{z}}(\underset{\sim}{t}) = \underbrace{e^{\mathbf{t}^T\mathbf{t}/2}}$ for all $\mathbf{t} \in \mathbb{R}^p$.

$\underset{\sim}{z} = (Z_1, \dots, Z_p)^T \Rightarrow M_{\underset{\sim}{z}}(\underset{\sim}{t}) = e^{\underset{\sim}{t}^T\underset{\sim}{t}/2}.$

$$M_{y}(t) = e^{t^T \mu + t^T V t/2}$$

**Exercise:** Let $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \mathbf{V})$ and $\cancel{\mathbf{y}}\ \mathbf{w} = \mathbf{a} + \mathbf{B}\mathbf{y}$. Show that

$$\mathbf{w}\ \cancel{\mathbf{y}} \sim \text{Normal}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\mathbf{V}\mathbf{B}^T).$$

$$M_{w}(t) = \mathbb{E}\, e^{t^T w} = \mathbb{E}\, e^{t^T(a + By)}$$

$$= e^{t^T a}\, \mathbb{E}\, e^{(B^T t)^T y}$$

$$= e^{t^T a}\, M_{y}(B^T t)$$

$$= e^{t^T a}\, e^{(B^T t)^T \mu + (B^T t)^T V (B^T t)/2}$$

$$= e^{t^T(a + B\mu) + t^T BVB^T t/2} \quad \leftarrow \text{mgf of}$$

## Result (Distributions of "blocks" of a Normal random vector)

*Let* $\mathbf{y} \sim$ *Normal*$(\boldsymbol{\mu}, \mathbf{V})$ *and partition* $\mathbf{y}$, $\boldsymbol{\mu}$, *and* $\mathbf{V}$ *as*

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix}}_{\sum p_j \times 1} \begin{matrix} \leftarrow p_1 \\ \\ \leftarrow p_K \end{matrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_K \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \dots & \mathbf{V}_{1K} \\ \vdots & \ddots \, \mathbf{V}_{jj} & \vdots \\ \mathbf{V}_{K1} & \dots & \mathbf{V}_{KK} \end{bmatrix}.$$

*Then:*

1. $\mathbf{y}_j \sim$ *Normal*$(\boldsymbol{\mu}_j, \mathbf{V}_{jj})$ *for each* $j$, $j = 1, \dots, K$.
2. $\mathbf{y}_1, \dots, \mathbf{y}_K$ *are mutually independent iff* $\mathbf{V}_{ij} = \mathbf{0}$ *for all* $i \neq j$.

See Cor 5.1 and Res 5.4 of Monahan (2008).

**Prove the result.**

① $\underset{p_j \times 1}{\underset{\sim}{y}_j} = \underset{\sim}{a} + B\underset{\sim}{y} \sim N(\underset{\sim}{a} + B\underset{\sim}{\mu}, \; BVB^T)$. Take $\underset{\sim}{a} = \underset{\sim}{0}$

$\underset{p_j \times \sum_k p_k)}{B} = [0 \cdots I_{p_j} \; 0 \cdots 0]$

Then $\quad B\underset{\sim}{\mu} = \underset{\sim}{\mu}_j \quad , \quad \underline{B V B^T} = V_{jj}$

$$B\underset{\sim}{\mu} = [0 \cdots 0 \;\; I_{p_j} \;\; 0 \cdots 0] \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \vdots \\ \underset{\sim}{\mu}_j \\ \vdots \\ \underset{\sim}{\mu}_k \end{bmatrix} = \underset{\sim}{\mu}_j$$

$$B V B^T = [0 \cdots 0 \;\; I_{p_j} \;\; 0 \cdots 0] \begin{bmatrix} V_{11} & & V_{1k} \\ & V_{jj} & \\ V_{k1} & & V_{kk} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{p_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= V_{jj} .$$

**Exercise:** Let $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \mathbf{V})$ and set

$$\mathbf{w}_1 = \mathbf{a}_1 + \mathbf{B}_1\mathbf{y} \quad \text{and} \quad \mathbf{w}_2 = \mathbf{a}_2 + \mathbf{B}_2\mathbf{y}.$$

Show that $\mathbf{w}_1$ and $\mathbf{w}_2$ are independent if and only if $\mathbf{B}_1\mathbf{V}\mathbf{B}_2^T = \mathbf{0}$.

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} y}_{a + B y} \sim N\left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mu, \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} V \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}^T}_{\begin{pmatrix} B_1 V B_1^T & B_1 V B_2^T \\ B_2 V B_1^T & B_2 V B_2^T \end{pmatrix}} \right)$$

$w_1$ and $w_2$ are indep. if this is zero.

These are orthogonal.

$$\hat{\underset{\sim}{y}} = P_X \underset{\sim}{y} \qquad \text{fitted values}$$

$$\hat{\underset{\sim}{e}} = (I - P_X)\underset{\sim}{y} \qquad \text{residuals}$$

$$\boxed{\hat{\underset{\sim}{y}} \cdot \hat{\underset{\sim}{e}} = 0}$$

**Exercise:** Let $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$, $\mathbf{e} \sim \text{Normal}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$.

1. Find the joint distribution of $\hat{\mathbf{y}} = \mathbf{P}_\mathbf{X}$ and $\hat{\mathbf{e}} = (\mathbf{I}_n - \mathbf{P}_\mathbf{X})\mathbf{y}$.
2. Check whether $\hat{\mathbf{y}}$ and $\hat{\mathbf{e}}$ are independent.

$$\begin{bmatrix} \hat{\underset{\sim}{y}} \\ \hat{\underset{\sim}{e}} \end{bmatrix} = \begin{bmatrix} P_X \underset{\sim}{y} \\ (I - P_X)\underset{\sim}{y} \end{bmatrix} = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} \underset{\sim}{y} \sim N\left( \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} \underset{\sim}{\mu}, \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} \sigma^2 I_n \begin{bmatrix} P_X \\ I - P_X \end{bmatrix}^T \right)$$

where

$$\begin{bmatrix} P_X \\ I - P_X \end{bmatrix} \sigma^2 I_n \begin{bmatrix} P_X \\ I - P_X \end{bmatrix}^T = \sigma^2 \begin{bmatrix} P_X P_X & P_X (I - P_X) \\ (I - P_X) P_X & (I - P_X)(I - P_X) \end{bmatrix} = \sigma^2 \begin{pmatrix} P_X & 0 \\ 0 & I - P_X \end{pmatrix}$$

$\Rightarrow$ $\hat{\underset{\sim}{y}}$ and $\hat{\underset{\sim}{e}}$ are independent

**Chi-squared distributions:**

- The pdf of the $\chi^2_\nu$ distribution is given by

$$f_W(w;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} w^{\nu/2-1} \exp\left[-\frac{w}{2}\right], \qquad \text{for } x > 0.$$

- $\nu$ is called the *degrees of freedom*
- mgf: $M_W(t) = (1-2t)^{-\nu/2}$ for $t < 1/2$.
- Let $\chi^2_{\nu,\xi}$ satisfy $P(W > \chi^2_{\nu,\xi}) = \xi$, where $W \sim \chi^2_\nu$.

Let $Z_1, \dots, Z_\nu \overset{\text{ind}}{\sim} \text{Normal}(0,1)$, then

$$W = Z_1^2 + \cdots + Z_\nu^2 \sim \chi^2_\nu.$$

**Exercise:** Prove the above using mgfs.

**t distributions:**

- The pdf of the $t_\nu$ distribution is given by

$$f_T(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{for } t \in \mathbb{R}.$$

- $\nu$ is called the *degrees of freedom*
- mgf: does not exist!
- Let $t_{\nu,\xi}$ satisfy $P(T > t_{\nu,\xi}) = \xi$, where $T \sim t_\nu$.

Let $Z$ and $W$ be independent rvs such that $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$, then

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu.$$

**Exercise:** Prove above via finding the joint density of $(T, U)$, where $U = W$.

## F distributions:

- The pdf of the $F_{\nu_1,\nu_2}$ distribution is given by

$$f_R(r;\nu_1,\nu_2)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} r^{(\nu_1-2)/2}\left(1+\frac{\nu_1}{\nu_2}r\right)^{-(\nu_1+\nu_2)/2}$$

  for $r>0$.
- $\nu_1$ and $\nu_2$ are called the *numerator and denominator degrees of freedom*.
- mgf: does not exist!
- Let $F_{\nu_1,\nu_2,\xi}$ satisfy $P(R>F_{\nu_1,\nu_2,\xi})=\xi$, where $R\sim F_{\nu_1,\nu_2,\xi}$.

Let $W_1$ and $W_2$ be independent rvs such that $W_1\sim\chi^2_{\nu_1}$ and $W_2\sim\chi^2_{\nu_2}$, then

$$R=\frac{W_1/\nu_1}{W_2/\nu_2}\sim F_{\nu_1,\nu_2}.$$

**Exercise:** Prove above via finding the joint density of $(R,U)$, where $U=W_2$.

## Non-central t distributions:

- The pdf of the $t_{\nu,\phi}$ distribution is given by

$$f_T(t;\nu,\phi) = \frac{e^{-\phi^2/2}}{\sqrt{\pi\nu}\Gamma(\nu/2)} \sum_{k=0}^{\infty} \frac{(2/\nu)^{k/2}(\phi t)^k}{k!} \frac{\Gamma((\nu+k+1)/2)}{(1+(t^2/\nu))^{(\nu+k+1)/2}}, \quad t \in \mathbb{R}.$$

- $\nu$ is called the *degrees of freedom*
- $\phi$ is called the *non-centrality parameter*
- mgf: does not exist!

For $Z \sim \text{Normal}(0,1)$ and $W \sim \chi^2_\nu$ with $Z \perp\!\!\!\perp W$, for any $\phi \in \mathbb{R}$, we have

$$T = \frac{Z+\phi}{\sqrt{W/\nu}} \sim t_{\nu,\phi}.$$

**Exercise:** For $X_1,\dots,X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu,\sigma^2)$, show that

$$\sqrt{n}(\bar{X}_n - \mu_0)/S_n \sim t_{n-1,\phi}, \quad \text{with } \phi = \sqrt{n}(\mu-\mu_0)/\sigma.$$

## Non-central chi-squared distributions:

- The pdf of the $\chi^2_\nu(\phi)$ distribution is given by

$$f_W(w;\nu,\phi) = \sum_{k=0}^{\infty} \frac{e^{-\phi/2}(\phi/2)^k}{k!} \frac{1}{\Gamma(\frac{\nu+2k}{2})2^{\frac{\nu+2k}{2}}} w^{\frac{\nu+2k}{2}-1} e^{-w/2}, \qquad \text{for } w > 0.$$

- $\nu$ is called the *degrees of freedom*.
- $\phi$ is called the *non-centrality parameter* (ncp).
- mgf: $M_W(t) = \exp(\phi t/(1-2t))(1-2t)^{-\nu/2}$ for $t < 1/2$.

Let $Z_1, \dots, Z_\nu \overset{\text{ind}}{\sim} \text{Normal}(0,1)$, then

$$W = (Z_1 + \mu_1)^2 + \cdots + (Z_\nu + \mu_\nu)^2 \sim \chi^2_\nu(\phi = \mu_1^2 + \cdots + \mu_\nu^2).$$

*Note that Monahan (2008) eq. (5.4) parameterizes the ncp differently.*

**Exercise:** Prove the above using mgfs.

## Non-central F distributions:

- The pdf of the $F_{\nu_1,\nu_2}(\phi)$ distribution is given by

$$f_R(r;\nu_1,\nu_2) = \sum_{k=0}^{\infty} \frac{e^{-\phi/2}(\phi/2)^2}{k!} \frac{\Gamma(\frac{\nu_1+\nu_2+2k}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_1+2k}{2})} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+2k}{2}} r^{\frac{\nu_1+2k}{2}-1}}{\left(1+r\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2+2k}{2}}}, \quad r > 0.$$

- $\nu_1$ and $\nu_2$ are called the *numerator and denominator degrees of freedom*.
- $\phi$ is the *non-centrality parameter*.
- mgf: does not exist!

Let $W_1$ and $W_2$ be independent rvs such that $W_1 \sim \chi^2_{\nu_1}(\phi)$ and $W_2 \sim \chi^2_{\nu_2}$, then

$$R = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1,\nu_2}(\phi).$$

**Exercise:** Prove above via finding the joint density of $(R, U)$, where $U = W_2$.

We will need the following result.

### Result (Spectral decomposition for an idempotent matrix)

*Let* $\mathbf{A}$ *be a symmetric* $p \times p$ *matrix. Then* $\mathbf{A}$ *is idempotent with rank* $s$ *if and only if there exists a* $p \times s$ *matrix* $\mathbf{G}$ *such that* $\mathbf{G}^T\mathbf{G} = \mathbf{I}_s$ *and* $\mathbf{G}\mathbf{G}^T = \mathbf{A}$.

**Prove the result.**

$A$ is symm, $p \times p$. Then

$A$ is idem w/ rank $s$ $\iff$ $\exists$ $\underset{p \times s}{G}$ s.t. $G^T G = I_s$

and $G G^T = A$.

$\Rightarrow$ Let $A$ idem w/ rank $\underline{s}$.

$$A = PDP^T \quad , \quad P^TP = I$$

$\uparrow$ diag w/ eigenvalues, $s$ are nonzero

$$= P_s D_s P_s^T$$

$\uparrow$ Identity, since eigenvalues are $0, 1$ (idempotent)

$$= P_s P_s^T$$

& $\underset{p \times s}{G} = P_s$ , $P_s^T P_s = I_s$

$\Leftarrow$ Suppose $\exists \underset{p \times s}{G}$ s.t. $G^TG = I_s$ and $GG^T = A$.

Then $AA = G\underbrace{G^T G}_{I_s}G^T = GG^T = A$.

Also $\text{rank}(A) = \text{rank}(GG^T) = \text{rank}(G) = s$.

$$\boxed{\text{Col } X^TX = \text{Col } X^T}$$

$$y_j = \mu_j + z_j\sigma, \quad z_j \sim N(0,1), \quad j=1,\dots,p,$$

$$\left[\ (z_1 + a_1)^2 + \dots + (z_p + a_p)^2 \sim \chi^2_p\left(\phi = \sum_{j=1}^p a_j^2\right)\right.$$

**Result (Chi-square results for quadratic forms in indep. Normals.)**

*Let* $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_p)$.

1. *We have* $\mathbf{y}^T\mathbf{y}/\sigma^2 \sim \chi^2_p(\phi = \boldsymbol{\mu}^T\boldsymbol{\mu}/\sigma^2)$.
2. *If* $\mathbf{A}$ *symm. idem. w/ rank s, then* $\mathbf{y}^T\mathbf{A}\mathbf{y}/\sigma^2 \sim \chi^2_s(\phi = \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}/\sigma^2)$

Cf. Res 5.9 and 5.14 Monahan (2008).

**Prove the results.**

(1) $$\frac{\underset{\sim}{y}^T\underset{\sim}{y}}{\sigma^2} = \frac{1}{\sigma^2}\sum_{j=1}^p y_j^2 = \frac{1}{\sigma^2}\sum_{j=1}^p(\mu_j + \sigma z_j)^2 = \sum_{j=1}^p \frac{\sigma^2\left(\frac{\mu_j}{\sigma} + z_j\right)^2}{\sigma^2}$$

$$= \sum_{j=1}^p\left(\frac{\mu_j}{\sigma} + z_j\right)^2 \sim \chi^2_p\left(\phi = \sum \frac{\mu_j^2}{\sigma^2}\right) = \chi^2_p\left(\phi = \frac{\underset{\sim}{\mu}^T\underset{\sim}{\mu}}{\sigma^2}\right)$$

$$\boxed{\underset{p\times p}{A} = \underset{p\times s}{G}\,\underset{s\times p}{G^T}, \quad G^T G = I_s}$$ because $A$ symm, idem.

② $$\frac{\underset{\sim}{y}^T A \underset{\sim}{y}}{\sigma^2} = \frac{\underset{\sim}{y}^T G G^T \underset{\sim}{y}}{\sigma^2} = \frac{(G^T \underset{\sim}{y})^T (G^T \underset{\sim}{y})}{\sigma^2}, \quad G^T \underset{\sim}{y} \sim \text{Normal}\left(G^T \underset{\sim}{\mu},\ \sigma^2 G^T G\right)$$

$$= \text{Normal}\left(G^T \mu,\ \sigma^2 I_s\right)$$

$$\sim \chi^2_s\left(\phi = \frac{(G^T \underset{\sim}{\mu})^T (G^T \mu)}{\sigma^2}\right)$$

$$= \chi^2_s\left(\phi = \frac{\underset{\sim}{\mu}^T G G^T \underset{\sim}{\mu}}{\sigma^2}\right)$$

$$= \chi^2_s\left(\phi = \frac{\underset{\sim}{\mu}^T A \underset{\sim}{\mu}}{\sigma^2}\right)$$

**Exercise:** Let $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, where $\mathbf{e} \sim \text{Normal}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ and $\mathbf{X}$ has rank $r$. Define

$$F = \frac{\|\hat{\mathbf{y}}\|^2 / r}{\|\hat{\mathbf{e}}\|^2/(n-r)},$$

where $\hat{\mathbf{y}} = \mathbf{P}_{\mathbf{X}}\mathbf{y}$ and $\hat{\mathbf{e}} = (\mathbf{I}_n - \mathbf{P}_{\mathbf{X}})\mathbf{y}$. Find the distribution of $F$.

$$F = \frac{\frac{1}{\sigma^2}\|\hat{\underset{\sim}{y}}\|^2/r}{\frac{1}{\sigma^2}\|\hat{\underset{\sim}{e}}\|^2/(n-r)} = \frac{\frac{1}{\sigma^2}\underset{\sim}{y}^T P_X \underset{\sim}{y} \;/r}{\frac{1}{\sigma^2}\underset{\sim}{y}^T(I-P_X)\underset{\sim}{y}\;/(n-r)} = \frac{\chi^2_r\left(\phi = \frac{\|X\underset{\sim}{b}\|^2}{\sigma^2}\right)/r}{\chi^2_{n-r}/(n-r)}$$

$$\sim F_{r,n-r}\left(\phi = \frac{\|X\underset{\sim}{b}\|^2}{\sigma^2}\right)$$

$$\frac{1}{\sigma^2}\underset{\sim}{y}^T P_X \underset{\sim}{y} \sim \chi^2_r\left(\phi = \frac{(X\underset{\sim}{b})^T P_X (X\underset{\sim}{b})}{\sigma^2}\right) = \chi^2_r\left(\phi = \frac{\|X\underset{\sim}{b}\|^2}{\sigma^2}\right)$$

2. *If* $\mathbf{A}$ *symm. idem. w/ rank* $s$, *then* $\mathbf{y}^T\mathbf{A}\mathbf{y}/\sigma^2 \sim \chi^2_s(\phi = \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}/\sigma^2)$

$$\frac{1}{\sigma^2}\, \underset{\sim}{y}^T (I - P_x)\, \underset{\sim}{y} \sim \chi^2_{n-r}\left(\phi = \underbrace{\frac{(x\underset{\sim}{b})^T (I - P_x) x\underset{\sim}{b}}{\sigma^2}}_{=0}\right)$$

$$= \chi^2_{n-r}$$

## Result (Chi-square results for quadratic forms in Normals.)

*Let* $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \mathbf{V})$, $\mathbf{V}$ *a* $p \times p$ *positive definite matrix.*

1. *We have* $\mathbf{y}^T\mathbf{V}^{-1}\mathbf{y} \sim \chi^2_p(\phi = \boldsymbol{\mu}^T\mathbf{V}^{-1}\boldsymbol{\mu})$
2. *If* $\mathbf{A}$ *is symm. and* $\mathbf{AV}$ *is idem. w/rank* $s$, *then* $\mathbf{y}^T\mathbf{A}\mathbf{y} \sim \chi^2_s(\phi = \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu})$.

Cf. Res 5.10 and 5.15 of Monahan (2008).

## Result (Independence of a linear function and a quadratic form)

*Let* $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \mathbf{V})$ *and* $\mathbf{A}$ *be symm. with rank s. If* $\boxed{\mathbf{BVA} = \mathbf{0}}$ *then* $\mathbf{By}$ *and* $\mathbf{y}^T\mathbf{Ay}$ *are independent.*

**Prove the result.**

$B\underset{\sim}{y}$

$\underset{\sim}{y}^T A \underset{\sim}{y}$ , $A$ symm. rank $s$.

Write $\quad A = P D P^T = P_s D_s P_s^T$

Then
$$\begin{bmatrix} B\underset{\sim}{y} \\ P_s^T \underset{\sim}{y} \end{bmatrix} = \begin{bmatrix} B \\ P_s^T \end{bmatrix} \underset{\sim}{y} \sim \text{Normal}\left( \begin{bmatrix} B \\ P_s^T \end{bmatrix} \underset{\sim}{\mu} , \underbrace{\begin{pmatrix} B \\ P_s^T \end{pmatrix} V \begin{pmatrix} B^T & P_s \end{pmatrix}} \right)$$

$$\begin{pmatrix} BVB^T & \underbrace{BVP_s}_{=0} \\ P_s^T V B^T & P_s^T V P_s \end{pmatrix}$$

We have

$$BVA = 0 \Rightarrow BVP_sD_sP_s^T = 0$$

$$\Rightarrow BVP_sD_s\underbrace{P_s^T\left(P_sD_s^{-1}\right)} = 0$$

$$\Rightarrow BVP_s = 0$$

So $B\underset{\sim}{y}$ and $P_s^T\underset{\sim}{y}$ are independent.

$\Rightarrow$ $B\underset{\sim}{y}$ and $D_s^{1/2}P_s^T\underset{\sim}{y}$ are independent

$\Rightarrow$ $B\underset{\sim}{y}$ and $\|D_s^{1/2}P_s^T\underset{\sim}{y}\|^2 = \underset{\sim}{y}^T P_s D_s^{1/2} D_s^{1/2} P_s^T \underset{\sim}{y}$

$$= \underset{\sim}{y}^T P_s D_s P_s^T \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T A \underset{\sim}{y}$$

are independent.

$$T_{stat} = \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{n}}$$

Under $H_0: \mu = \mu_0$, $\frac{\bar{x}_n - \mu_0}{s_n/\sqrt{n}} \sim t_{n-1}$

$$\frac{Z}{\sqrt{W/\nu}} \sim t_\nu \qquad Z \sim N(0,1), \quad W \sim \chi^2_\nu, \quad Z \perp\!\!\!\perp W$$

**Exercise:** Let $X_1, \dots, X_n \overset{\text{ind}}{\sim} \text{Normal}(\mu, \sigma^2)$ and let

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 \quad \text{and} \quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i.$$

Show that $S_n^2$ and $\bar{X}_n$ are independent.

$$\underset{\sim}{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \sim \text{Normal}(\underset{\sim}{\mu}, \sigma^2 I_n).$$

$$V = \sigma^2 I_n$$

$$B\underset{\sim}{x} = \bar{x}_n \qquad B = \frac{1}{n}\underset{\sim}{1}_n^T.$$

$$(n-1)S_n^2 = \underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T\left(I - \frac{1}{n}\underset{\sim}{1}_n\underset{\sim}{1}_n^T\right)\underset{\sim}{x}$$

$$\left(I - \frac{1}{n}\underset{\sim}{1}_n\underset{\sim}{1}_n^T\right)\underset{\sim}{1}_n$$

$$\begin{aligned}
BVA &= \frac{1}{n}\underset{\sim}{1}_n^T \sigma^2 I_n \left(I_n - \frac{1}{n}\underset{\sim}{1}_n\underset{\sim}{1}_n^T\right) \\
&= \sigma^2\frac{1}{n}\left(\underset{\sim}{1}_n^T - \frac{1}{n}\underbrace{\underset{\sim}{1}_n^T\underset{\sim}{1}_n}_{n}\underset{\sim}{1}_n^T\right) \\
&= \sigma^2\frac{1}{n}\left(\underset{\sim}{1}_n^T - \underset{\sim}{1}_n^T\right) \\
&= \underset{\sim}{0}^T
\end{aligned}$$

## Theorem (Cochran)

*Let* $\mathbf{y} \sim \text{Normal}(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_n)$ *and let* $\mathbf{A}_1, \dots, \mathbf{A}_K$ *be symmetric, idempotent matrices with ranks* $s_1, \dots, s_K$ *such that* $\sum_{k=1}^K \mathbf{A}_k = \mathbf{I}_n$*. Then*

$$\frac{\mathbf{y}^T \mathbf{A}_1 \mathbf{y}}{\sigma^2}, \dots, \frac{\mathbf{y}^T \mathbf{A}_K \mathbf{y}}{\sigma^2} \quad \textit{are independent}$$

*and* $\dfrac{\mathbf{y}^T \mathbf{A}_k \mathbf{y}}{\sigma^2} \sim \chi^2_{s_k}\left(\phi = \dfrac{\boldsymbol{\mu}^T \mathbf{A}_k \boldsymbol{\mu}}{\sigma^2}\right)$ *for each* $k = 1, \dots, K$.

**Prove the result.**

$$\boxed{\underset{n\times n}{A_k} = \underset{n\times s_k}{G_k}\,\underset{s_k\times n}{G_k^T}, \quad G_k^T G_k = I_{s_k} \quad \text{for } k=1,\dots,K.}$$

$$\frac{\tilde{y}^T A_k \tilde{y}}{\sigma^2} = \frac{\tilde{y}^T G_k G_k^T \tilde{y}}{\sigma^2} = \frac{(G_k^T \tilde{y})^T G_k^T \tilde{y}}{\sigma^2}$$

Sufficient to show $G_1^T \underset{\sim}{y}, \ldots, G_K^T \underset{\sim}{y}$ are indep.

Write $G = [G_1 \cdots G_K]$.

Then

$$G^T \underset{\sim}{y} = \begin{bmatrix} G_1^T \underset{\sim}{y} \\ \vdots \\ G_K^T \underset{\sim}{y} \end{bmatrix} \sim N\left( \begin{bmatrix} G_1^T \\ \vdots \\ G_K^T \end{bmatrix} \underset{\sim}{\mu},\ \underbrace{G^T (\sigma I_n) G}_{\underbrace{\sigma^2 \begin{pmatrix} G_1^T \\ \vdots \\ G_K^T \end{pmatrix} (G_1 \cdots G_K)}_{\underbrace{\sigma^2 \begin{pmatrix} G_1^T G_1 & \cdots & G_1^T G_K \\ & & \\ G_K^T G_1 & \cdots & G_K^T G_K \end{pmatrix}}_{I}}} \right)$$

$$= \sigma^2 G^T G.$$

Need to show that $G^T G$ is the identity matrix.

We have

$$G G^T = [G_1 \cdots G_K] \begin{bmatrix} G_1^T \\ \vdots \\ G_K^T \end{bmatrix} = \sum_{k=1}^{K} G_k G_k^T = \sum_{k=1}^{K} A_k = I_n.$$

Take

$$\operatorname{tr}(G G^T) = n \implies \operatorname{tr}(G^T G) = n,$$

where $\operatorname{tr}(G^T G) = \sum_{k=1}^{K} \operatorname{tr}(G_k^T G_k) = \sum_{k=1}^{K} \operatorname{tr}(I_{s_k}) = \sum_{k=1}^{K} s_k = n$

The fact $\sum_{k=1}^{K} s_k = n$ tells us that $G = [\underset{n \times s_1}{G_1} \cdots \underset{n \times s_K}{G_K}]$ is $n \times n$.

So $G$ is square and $G G^T = I_n$. This gives $G^T G = I_n$.

## Result (Sequential sums of squares by Cochran's theorem)

*Let* $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, *where* $\mathbf{e} \sim \mathrm{Normal}(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_n)$ *and* $\mathbf{X} = [\mathbf{X}_0\ \mathbf{X}_1\ \ \ldots\ \mathbf{X}_K]$. *Define*

$$\mathbf{P}_0 = \mathbf{P}_{\mathbf{X}_0}, \quad \mathbf{P}_1 = \mathbf{P}_{[\mathbf{X}_0\ \mathbf{X}_1]}, \quad \mathbf{P}_2 = \mathbf{P}_{[\mathbf{X}_0\ \mathbf{X}_1\ \mathbf{X}_2]}, \quad \ldots \quad \mathbf{P}_K = \mathbf{P}_{\mathbf{X}}$$

*and set* $s_0 = \operatorname{rank} \mathbf{P}_0$, $s_k = \operatorname{rank}(\mathbf{P}_k - \mathbf{P}_{k-1})$, $k = 1, \ldots, K$. *Then we have*

$$\frac{1}{\sigma^2}\mathbf{y}^T\mathbf{P}_0\mathbf{y} \sim \chi^2_{s_0}\left(\phi = \frac{1}{\sigma^2}(\mathbf{X}\mathbf{b})^T\mathbf{P}_0\mathbf{X}\mathbf{b}\right)$$

$$\frac{1}{\sigma^2}\mathbf{y}^T(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{y} \sim \chi^2_{s_k}\left(\phi = \frac{1}{\sigma^2}(\mathbf{X}\mathbf{b})^T(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{X}\mathbf{b}\right), \quad k = 1, \ldots, K$$

$$\frac{1}{\sigma^2}\mathbf{y}^T(\mathbf{I} - \mathbf{P}_K)\mathbf{y} \sim \chi^2_{n - \sum_{k=1}^K s_k},$$

*where the above quantities are mutually independent.*

**Show that this is an application of Cochran's theorem.**

We can summarize the sequential sums of squares and their distribution in a table:

| Source | $\text{SS}/\sigma^2$ | df | $\phi$ |
|---|---|---|---|
| $\mathbf{X}_0$ | $\frac{1}{\sigma^2}\mathbf{y}^T\mathbf{P}_0\mathbf{y}$ | $s_0$ | $\frac{1}{\sigma^2}(\mathbf{X}\mathbf{b})^T\mathbf{P}_0(\mathbf{X}\mathbf{b})$ |
| $\mathbf{X}_k$ | $\frac{1}{\sigma^2}\mathbf{y}^T(\mathbf{P}_k - \mathbf{P}_{k-1})\mathbf{y}$ | $s_k$ | $\frac{1}{\sigma^2}(\mathbf{X}\mathbf{b})^T(\mathbf{P}_k - \mathbf{P}_{k-1})(\mathbf{X}\mathbf{b})$ |
| $\mathbf{e}$ | $\frac{1}{\sigma^2}\mathbf{y}^T(\mathbf{I} - \mathbf{P}_K)\mathbf{y}$ | $n - \sum_{k=0}^K s_k$ | 0 |

$$X = \begin{bmatrix} X_0 & X_1 \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & & \\ \vdots & & \ddots & \\ \mathbf{1}_{n_a} & & & \mathbf{1}_{n_a} \end{bmatrix} \qquad b = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix} \qquad Y = Xb + e$$

$e \sim N(0, \sigma^2 I_N)$

$n_\cdot = \sum_{i=1} n_i$

**Exercise:** Let $Y_{ij} = \boxed{\mu + \alpha_i} + \varepsilon_{ij}$, $\varepsilon_{ij} \overset{\text{ind}}{\sim} \text{Normal}(0, \sigma^2)$, $i = 1, \dots, a$, $j = 1, \dots, n_i$.

Derive the ANOVA table:

| Source | SS | df | $\phi$ |
|---|---|---|---|
| Mean | $n_\cdot(\bar{y}_{\cdot\cdot})^2$ | 1 | $n_\cdot\left(\mu + n_\cdot^{-1}\sum_{i=1}^a n_i\alpha_i\right)^2/\sigma^2$ |
| Treatment | $\sum_{i=1}^a n_i(\bar{y}_i - \bar{y}_{\cdot\cdot})^2$ | $a-1$ | $\sum_{i=1}^a n_i(\alpha_i - n_\cdot^{-1}\sum_{j=1}^a n_j\alpha_j)^2/\sigma^2$ |
| Error | $\sum_{i=1}^a\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i\cdot})^2$ | $n_\cdot - a$ | 0 |

| | | | |
|---|---|---|---|
| mean | $Y^T P_0 Y$ | rank $P_0$ | $(Xb)^T P_0 Xb$ |
| Treatment | $Y^T(P_1 - P_0)Y$ | rank $(P_1 - P_0)$ | $(Xb)^T(P_1 - P_0)Xb$ |
| Error | $Y^T(I - P_1)Y$ | rank $(I - P_1)$ | $(Xb)^T(I - P_1)Xb$ |

$\underset{\sim}{y}^T P_0 \underset{\sim}{y}$ :

$$P_0 \underset{\sim}{y} = \underset{\sim}{1}_{n.}(\underset{\sim}{1}_{n.}^T \underset{\sim}{1}_{n.})^{-1} \underset{\sim}{1}_{n.}^T \underset{\sim}{y} = \underset{\sim}{1}_{n.} \frac{1}{n.} \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij} = \underset{\sim}{1}_{n.} \bar{y}_{..}$$

$$\underset{\sim}{y}^T P_0 \underset{\sim}{y} = \| P_0 \underset{\sim}{y} \|^2 = n. (\bar{y}_{..})^2$$

$(X\underset{\sim}{b})^T P_0 X\underset{\sim}{b}$ :

$$P_0 X\underset{\sim}{b} = \underset{\sim}{1}_{n.} (\underset{\sim}{1}_{n.}^T \underset{\sim}{1}_{n.})^{-1} \underset{\sim}{1}_{n.}^T (X\underset{\sim}{b})$$

$$= \underset{\sim}{1}_{n.} \frac{1}{n.} [n_1(\mu+\alpha_1) + n_2(\mu+\alpha_2) + \dots + n_a(\mu+\alpha_a)]$$

$$X\underset{\sim}{b} = \begin{bmatrix} \mu+\alpha_1 \\ \vdots \\ \mu+\alpha_1 \\ \mu+\alpha_2 \\ \vdots \\ \mu+\alpha_2 \\ \vdots \\ \mu+\alpha_a \\ \vdots \\ \mu+\alpha_a \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n_2 \\ \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n_a \end{matrix}$$

$$= \underset{\sim}{1}_{n.} \frac{1}{n.} \sum_{i=1}^{a} n_i(\mu+\alpha_i)$$

$$= \underset{\sim}{1}_{n.} \left(\mu + \frac{1}{n.} \sum_{i=1}^{a} n_i \alpha_i\right)$$

$$\Rightarrow (X\underset{\sim}{b})^T P_0 X\underset{\sim}{b} = \| P_0 X\underset{\sim}{b} \|^2$$

$$= n. \left(\mu + \frac{1}{n.} \sum_{i=1}^{a} n_i \alpha_i\right)^2$$

$\underset{\sim}{y}^T (P_1 - P_0) \underset{\sim}{y}$ :

$$(P_1 - P_0)\underset{\sim}{y}$$

$$P_1 \underset{\sim}{y} : P_X = P_{X_1} = X_1 (X_1^T X_1)^{-1} X_1^T$$

$$= X_1 \begin{bmatrix} n_1 & & \\ & \ddots & \\ & & n_a \end{bmatrix}^{-1} X_1^T$$

$$\Rightarrow P_{X_1} \underset{\sim}{y} = X_1 \begin{bmatrix} \frac{1}{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_a} \end{bmatrix} X_1^T \underset{\sim}{y}$$

$$= X_1 \begin{bmatrix} \frac{1}{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_a} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{n_1}^T & & \\ & \ddots & \\ & & \underset{\sim}{1}_{n_a}^T \end{bmatrix} \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \vdots \\ y_{a1} \\ \vdots \\ y_{an_a} \end{bmatrix}$$

$$= X_1 \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \\ \vdots \\ \bar{y}_{a\cdot} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{1}_{n_1} \bar{y}_{1\cdot} \\ \vdots \\ \underset{\sim}{1}_{n_a} \bar{y}_{a\cdot} \end{bmatrix}$$

$$(P_1 - P_0)\underset{\sim}{y} = (P_1 \underset{\sim}{y} - P_0 \underset{\sim}{y}) = \begin{bmatrix} \underset{\sim}{1}_{n_1} \bar{y}_{1\cdot} \\ \vdots \\ \underset{\sim}{1}_{n_a} \bar{y}_{a\cdot} \end{bmatrix} - \underset{\sim}{1}_{n\cdot} \bar{y}_{\cdot\cdot}$$

$$= \begin{bmatrix} \underset{\sim}{1}_{n_1} (\bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot}) \\ \vdots \\ \underset{\sim}{1}_{n_a} (\bar{y}_{a\cdot} - \bar{y}_{\cdot\cdot}) \end{bmatrix}$$

$$\underset{\sim}{y}^T (P_1 - P_0) \underset{\sim}{y} = \| (P_1 - P_0) \underset{\sim}{y} \|^2 = n_1 (\bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot})^2 + \dots + n_a (\bar{y}_{a\cdot} - \bar{y}_{\cdot\cdot})^2$$

$$= \sum_{i=1}^{a} n_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$$

$$(X\underset{\sim}{b}) (P_1 - P_0) X\underset{\sim}{b} = \qquad P_1 X \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{1}_{n_1} (\mu + \alpha_1) \\ \vdots \\ \underset{\sim}{1}_{n_a} (\mu + \alpha_a) \end{bmatrix}$$

$$P_0 X \underset{\sim}{b} = \underset{\sim}{1}_{n\cdot} \left( \mu + \frac{1}{n\cdot} \sum_{i=1}^{a} n_i \alpha_i \right)$$

$$(P_1 - P_0)X\underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{1}_{n_1}\left(\mu + \alpha_1 - \left(\mu + \frac{1}{n_\cdot}\sum_{i=1}^{a} n_i \alpha_i\right)\right) \\ \vdots \\ \underset{\sim}{1}_{n_a}\left(\mu + \alpha_a - \left(\mu + \frac{1}{n_\cdot}\sum_{i=1}^{a} n_i \alpha_i\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{1}_{n_1}\left(\alpha_1 - \frac{1}{n_\cdot}\sum n_i \alpha_i\right) \\ \vdots \\ \underset{\sim}{1}_{n_a}\left(\alpha_a - \frac{1}{n_\cdot}\sum n_i \alpha_i\right) \end{bmatrix}$$

Monahan, J. F. (2008). *A primer on linear models*. CRC Press.

$$(X\underset{\sim}{b})^T (P_1 - P_0) X\underset{\sim}{b} = \|(P_1 - P_0)X\underset{\sim}{b}\|^2 = \sum_{i=1}^{a} n_i \left(\alpha_i - \frac{1}{n_\cdot}\sum_{i=1}^{a} n_i \alpha_i\right)^2$$

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$$\underset{\sim}{y}^T (I - P_1)\underset{\sim}{y} = \|\underset{\sim}{y} - P_1 \underset{\sim}{y}\|^2 = \left\| \underset{\sim}{y} - \begin{bmatrix} \underset{\sim}{1}_{n_1}\bar{Y}_{1\cdot} \\ \vdots \\ \underset{\sim}{1}_{n_a}\bar{Y}_{a\cdot} \end{bmatrix} \right\|^2 = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\cdot})^2$$

$$X\underset{\sim}{b}(I - P_1)X\underset{\sim}{b} = 0$$

STAT 414 Thanksgiving

Thursday, Nov 23

12:00 noon at

Prof. Gregory's house

(Will send address in email - RSVP)

Bring a dish to share ☺

Prof. G will provide American Thxgiving staples.

Spouse, g-friend, b-friend, best friend also invited.