STAT 714 fa 2023 Exam 1

1. Let $Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$ for $i = 1, 2$, $j = 1, 2$, and $k = 1, \ldots, n_{ij}$. Assume $\mathbb{E}\varepsilon_{ijk} = 0$ for all i, j, k.
   (a) Put the model in matrix form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$. Write down $\mathbf{X}$ and $\mathbf{b}$ clearly.
   (b) Give the dimension of $\operatorname{Col}\mathbf{X}$ as well as the dimension of $\operatorname{Nul}\mathbf{X}$.
   (c) Give the dimension of $(\operatorname{Col}\mathbf{X})^{\perp}$.
   (d) Check whether the following contrasts are estimable in the model $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$:
      i. $\mu + \alpha_1$
      ii. $\alpha_2 - \alpha_1$
   (e) Let

$$\mathbf{W} = \begin{bmatrix} \mathbf{1}_{n_{11}} & \mathbf{1}_{n_{11}} & \mathbf{1}_{n_{11}} \\ \mathbf{1}_{n_{12}} & \mathbf{1}_{n_{12}} & \mathbf{0} \\ \mathbf{1}_{n_{21}} & \mathbf{0} & \mathbf{1}_{n_{21}} \\ \mathbf{1}_{n_{22}} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

   Show that $\mathbf{y} = \mathbf{W}\mathbf{d} + \mathbf{e}$ is a reparameterization of the model $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ by giving the matrices $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{W} = \mathbf{X}\mathbf{T}$ and $\mathbf{X} = \mathbf{W}\mathbf{S}$, which shows $\operatorname{Col}\mathbf{W} = \operatorname{Col}\mathbf{X}$.
   (f) Give the entries of $\mathbf{d} = \mathbf{S}\mathbf{b}$ in terms of the parameters μ, α_1, α_2, β_1, and β_2.
   (g) Argue carefully that *all* contrasts $\mathbf{c}^T\mathbf{d}$ for $\mathbf{c} \in \mathbb{R}^3$ are estimable in the model $\mathbf{y} = \mathbf{W}\mathbf{d} + \mathbf{e}$.
   (h) Give $\mathbf{W}^T\mathbf{W}$ as well as $\mathbf{W}^T\mathbf{y}$. Let $\bar{y}_{ij} = n_{ij}^{-1}\sum_{k=1}^{n_{ij}} Y_{ijk}$ for $i = 1, 2$ and $j = 1, 2$.
   (i) Give $\operatorname{tr}(\mathbf{P}_{\mathbf{W}})$ as well as $\operatorname{tr}(\mathbf{P}_{\mathbf{X}})$, where $\mathbf{P}_{\mathbf{W}} = \mathbf{W}(\mathbf{W}^T\mathbf{W})^{-}\mathbf{W}^T$ and $\mathbf{P}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T$.

2. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.
   (a) Give a basis for $\operatorname{Col}\mathbf{A}$.
   (b) Find a nontrivial solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$.
   (c) Find a vector which is orthogonal to every column of $\mathbf{A}$.

3. State whether each statement is true or false and give a proof supporting your answer.
   (a) For any matrix $\mathbf{X}$, the matrix $\mathbf{X}^T\mathbf{X}$ is positive definite.
   (b) If $\mathbf{A}$ is an invertible matrix and λ is an eigenvalue of $\mathbf{A}$ then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.