STAT 714 fa 2023 Exam I solutions

$\boxed{1}$ (a)

$$X = \begin{bmatrix} \underset{\sim}{1}_{n_{11}} & \underset{\sim}{1}_{n_{11}} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{11}} & \underset{\sim}{0} \\ \underset{\sim}{1}_{n_{12}} & \underset{\sim}{1}_{n_{12}} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{12}} \\ \underset{\sim}{1}_{n_{21}} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{21}} & \underset{\sim}{1}_{n_{21}} & \underset{\sim}{0} \\ \underset{\sim}{1}_{n_{22}} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{22}} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{22}} \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

(b) $\dim \operatorname{Col} X = 3$, $\dim \operatorname{Nul} X = 2$

(c) We have $(\operatorname{Col} X)^\perp = \operatorname{Nul} X^T$, and $n_{11}+n_{12}+n_{21}+n_{22} = \underbrace{\dim \operatorname{Col} X^T}_{3} + \dim \operatorname{Nul} X^T$

so $\dim (\operatorname{Col} X)^\perp = \dim \operatorname{Nul} X^T = n_{11}+n_{12}+n_{21}+n_{22} - 3$.

(d) (i) $\mu + \alpha_1 = \underset{\sim}{c}^T \underset{\sim}{b}$, with $\underset{\sim}{c}^T = [1 \ 1 \ 0 \ 0 \ 0]$

This is NOT estimable because $\underset{\sim}{c} \notin \operatorname{Col} X^T$.

To see this, note that $\operatorname{Col} X^T = \operatorname{Col} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Then write $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & \ddots & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$.

By row-reduction we see that $\underset{\sim}{c} \notin \operatorname{Col} X^T$.

(ii) $\alpha_2 - \alpha_1 = \underset{\sim}{c}^T \underset{\sim}{b}$ with $\underset{\sim}{c}^T = [0 \ -1 \ 1 \ 0 \ 0]$

This IS estimable because $\underset{\sim}{c} \in \operatorname{Col} X^T$.

We can see that $\operatorname{Col}_3(X^T) + \operatorname{Col}_4(X^T) - (\operatorname{Col}_1(X^T) + \operatorname{Col}_2(X^T)) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(e)

$$W = \begin{bmatrix} \underset{\sim}{1}_{n_{11}} & \underset{\sim}{1}_{n_{11}} & \underset{\sim}{1}_{n_{11}} \\ \underset{\sim}{1}_{n_{12}} & \underset{\sim}{1}_{n_{12}} & \underset{\sim}{0} \\ \underset{\sim}{1}_{n_{21}} & \underset{\sim}{0} & \underset{\sim}{1}_{n_{21}} \\ \underset{\sim}{1}_{n_{22}} & \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$$

We have

$$\underset{n\times 3}{W} = \underset{n\times 5}{X} \underbrace{\underset{5\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}}_{T} \quad \text{and} \quad \underset{n\times 5}{X} = \underset{n\times 3}{W} \underbrace{\underset{3\times 5}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}}}_{S}$$

(f)

$$\underset{\sim}{d} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mu + \alpha_1 + \beta_1 \\ \alpha_1 - \alpha_2 \\ \beta_1 - \beta_2 \end{bmatrix}$$

(g) We have $\operatorname{Col} W^T \subset \mathbb{R}^3$ and $\dim \operatorname{Col} W^T = 3$.

Therefore $\operatorname{Col} W^T = \mathbb{R}^3$.

The "$p > n$ theorem" tells us this: If I have 3 linearly independent vectors in $\mathbb{R}^3$, then adding any additional vector from $\mathbb{R}^3$ to the set will make it linearly dependent. This means every vector in $\mathbb{R}^3$ lies in the span of the set of the three linearly independent vectors I started with.

So $\underset{\sim}{c} \in \operatorname{Col} W^T$ for all $\mathbb{R}^3$.

(h)

$$W^T W = \begin{bmatrix} n_{11}+n_{12}+n_{21}+n_{22} & n_{11}+n_{12} & n_{11}+n_{21} \\ n_{11}+n_{12} & n_{11}+n_{12} & n_{11} \\ n_{11}+n_{21} & n_{11} & n_{11}+n_{21} \end{bmatrix}$$

$$W^T \underset{\sim}{y} = \begin{bmatrix} n_{11}\bar{y}_{11} + n_{12}\bar{y}_{12} + n_{21}\bar{y}_{21} + n_{22}\bar{y}_{22} \\ n_{11}\bar{y}_{11} + n_{12}\bar{y}_{12} \\ n_{11}\bar{y}_{11} + n_{21}\bar{y}_{21} \end{bmatrix}$$

(i) Since rank $W = 3$, we have $tr\left(W(W^TW)^- W^T\right) = 3$.

It is the same for X: rank $X = 3 \Rightarrow tr\left(X(X^TX)^- X^T\right) = 3$

$\boxed{2}$ Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Give a basis for Col A.

The first column is the sum of the second two columns, and the second two columns are linearly independent, so A has rank 2.

We can take

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

or any set of 2 of the columns as a basis for Col A.

(b) Find a nontrivial solution to $A\underset{\sim}{x} = \underset{\sim}{0}$.

We already noted that the first column is the sum of the other columns, so

$$A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \underset{\sim}{0}.$$

(c) Find a vector which is orthogonal to every column of A.

We can take any vector in $(\text{Col } A)^\perp$.

We have $(\text{Col } A)^\perp = \text{Nul } A^T = \text{Nul} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

Row-reduce

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + \tfrac{1}{2} x_3 = 0$$

$$-2x_2 + x_3 = 0 \qquad x_3 \text{ free}$$

This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \quad \text{for } x_3 \in \mathbb{R}.$$

$\leftarrow$ can scale by any constant.

So $$(\mathrm{Col}\, A)^{\perp} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

So $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to every column of A.

$\boxed{3}$ (a) **FALSE**: If X is not full-rank then $\mathrm{Nul}\, X = \mathrm{Nul}\, X^T X$ has positive dimension, so one can find a nonzero vector $\underset{\sim}{v}$ such that $X\underset{\sim}{v} = 0$ and therefore $\underset{\sim}{v}^T X^T X \underset{\sim}{v} = \underset{\sim}{0}$.

Since $\underset{\sim}{v}^T X^T X \underset{\sim}{v} = \|X\underset{\sim}{v}\| \geq 0$ for all $\underset{\sim}{v}$, however, $X^T X$ is positive semi-definite.

(b) **TRUE**: Let $\underset{\sim}{x}$ be a vector such that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

Then

$$A^{-1} A \underset{\sim}{x} = \lambda A^{-1} \underset{\sim}{x} \quad \Rightarrow \quad A^{-1} \underset{\sim}{x} = \frac{1}{\lambda} \underset{\sim}{x},$$

so $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.