STAT 714 fa 2023 Exam 02 solutions

$\boxed{1}$ (a) We have $\hat{\mu}_{gls} = (X^T V^{-1} X)^{-1} X^T V^{-1} \underset{\sim}{Y}$, where $X = \underset{\sim}{1}_n$ and $V = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$.

Since

$$X^T V^{-1} X = \underset{\sim}{1}_n^T \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{pmatrix} \underset{\sim}{1}_n = \sum_{i=1}^n \frac{1}{\sigma_i^2}$$

and

$$X^T V^{-1} \underset{\sim}{Y} = \underset{\sim}{1}_n^T \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{pmatrix} \underset{\sim}{Y} = \sum_{i=1}^n \frac{Y_i}{\sigma_i^2},$$

we obtain

$$\hat{\mu}_{gls} = \frac{\sum_{i=1}^n Y_i/\sigma_i^2}{\sum_{i=1}^n 1/\sigma_i^2}.$$

(b) We have $\hat{\mu}_{ols} = (X^T X)^{-1} X^T \underset{\sim}{Y} = (\underset{\sim}{1}_n^T \underset{\sim}{1}_n)^{-1} \underset{\sim}{1}_n^T \underset{\sim}{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$, so

$$\hat{\mu}_{ols} = \bar{Y}.$$

(c) We have

$$\operatorname{Var} \hat{\mu}_{gls} = \operatorname{Var}\left(\frac{\sum_{i=1}^n Y_i/\sigma_i^2}{\sum_{i=1}^n 1/\sigma_i^2} \right)$$

$$= \frac{1}{\left(\sum_{i=1}^n 1/\sigma_i^2\right)^2} \sum_{i=1}^n \frac{\overbrace{\operatorname{Var} Y_i}^{\sigma_i^2}}{\sigma_i^4}$$

$$= \frac{1}{\sum_{i=1}^n 1/\sigma_i^2}.$$

(d) We have $\operatorname{Var} \hat{\mu}_{ols} = \operatorname{Var} \bar{Y} = \frac{1}{n^2} \operatorname{Var}\left(\sum_{i=1}^n Y_i\right) = \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2.$

(e) If $\sigma_i^2 = \sigma^2$ for all i then

$$\text{Var } \hat{\mu}_{\text{gls}} = \frac{\sigma^2}{n} \qquad \text{and} \qquad \text{Var } \hat{\mu}_{\text{ols}} = \frac{\sigma^2}{n},$$

so the two become equal.

(f) We have

$$\sum_{i=1}^{n} \left(\frac{Y_i^2}{\sigma_i^2} \right) = \sum_{i=1}^{n} \left(\frac{\mu + \varepsilon_i}{\sigma_i} \right)^2$$

$$= \sum_{i=1}^{n} \left(\frac{\mu}{\sigma_i} + Z_i \right)^2, \quad \text{where } Z_i = \frac{\varepsilon_i}{\sigma_i}$$

$$\sim \chi_n^2 \left(\phi = \sum_{i=1}^{n} \frac{\mu^2}{\sigma_i^2} \right).$$

So for $\mu = 0$, $\displaystyle\sum_{i=1}^{n} \frac{Y_i^2}{\sigma_i^2} \sim \chi_n^2$.

$\boxed{2}$ (a)

$$X = \begin{bmatrix} 1 & & & x_{11} & & \\ \vdots & & & \vdots & & \\ 1 & & & x_{1n} & & \\ & 1 & & & x_{21} & \\ & \vdots & & & \vdots & \\ & 1 & & & x_{2n} & \\ & & 1 & & & x_{31} \\ & & \vdots & & & \vdots \\ & & 1 & & & x_{3n} \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

(b) We need $n \geq 2$. If $n = 1$ the columns will be linearly dependent.

(c) The assumption, along with requiring $n \geq 2$, gives X full column rank. This makes each of the parameters estimable.

Intuitively, we could not estimate the "slope" parameters $\beta_1, \beta_2, \beta_3$ if in a group we observed only a single (x, y) pair.

(d) Reformulate the hypothesis as $H_0: \beta_1 = 0$ & $\beta_2 = 0$ & $\beta_3 = 0$.

Then set $K^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $\underset{\sim}{m} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(e) The numerator df is rank $K = 3$.

The denominator df is the total # obs minus rank X. This is

$$3n - 6$$

(f) This corresponds to testing $H_0: \mu_1 - \mu_2 = 0$ & $\mu_2 - \mu_3 = 0$ & $\beta_1 - \beta_2 = 0$ & $\beta_2 - \beta_3 = 0$.

Set $K^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ and $\underset{\sim}{m} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(g) Numerator df = rank $K = 4$ and denominator df $= 3n - 6$.

$\boxed{3}$ (a) Let $\tilde{P}_X = X(X^T V^{-1} X)^- X^T V^{-1}$.

(i) We have $X \underbrace{(X^T V^{-1} X)^- X^T V^{-1}}_{g\text{ inv. of } X} X (X^T V^{-1} X)^- X^T V^{-1} = X (X^T V^{-1} X)^- X^T V^{-1}$, so $\tilde{P}_X$ is idempotent.

(ii) For any $\underset{\sim}{Y}$, $X (X^T V^{-1} X)^- X^T V^{-1} \underset{\sim}{Y} \in \text{Col}\, X$.

(iii) For $\underset{\sim}{z} \in \text{Col}\, X$, $\underset{\sim}{z} = X \underset{\sim}{a}$ for some $\underset{\sim}{a}$, so

$$X (X^T V^{-1} X)^- X^T V^{-1} \underset{\sim}{z} = X (X^T V^{-1} X)^- X^T V^{-1} X \underset{\sim}{a} = X \underset{\sim}{a} = \underset{\sim}{z}$$

(b) For any $\underset{\sim}{y} \in \mathbb{R}^n$, check whether

$$\underset{\sim}{y} = \tilde{P}_X \underset{\sim}{y} + (I - \tilde{P}_X)\underset{\sim}{y}$$

is an orthogonal decomposition.

We have

$$\left(\tilde{P}_X \underset{\sim}{y}\right) \cdot (I - \tilde{P}_X)\underset{\sim}{y} = \underset{\sim}{y}^T \tilde{P}_X^T (I - \tilde{P}_X)\underset{\sim}{y}$$

$$= \underset{\sim}{y}^T \left[V^{-1}X(X^TV^{-1}X)^- X^T \left(I - X(X^TV^{-1}X)^- X^TV^{-1}\right)\right] \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T \left[V^{-1}X(X^TV^{-1}X)^- X^T - V^{-1}X(X^TV^{-1}X)^- X^TX(X^TV^{-1}X)^- X^TV^{-1}\right] \underset{\sim}{y}$$

We see that this is 0 if $V = I\sigma^2$ for any σ^2, but it is not in general equal to 0.

So $\tilde{P}_X$ is not an orthogonal projection.

One can also invoke a result which says that a projection matrix is the matrix of an orthogonal projection <u>iff</u> it is symmetric.

Since $\tilde{P}_X$ is not symmetric, it is not an orthogonal projection.