STAT 714 fa 2023 final exam solutions

1 $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(a) Note that $\text{Col}_1(A) = \text{Col}_2(A) + \text{Col}_3(A)$.
Also $\text{Col}_2(A)$ and $\text{Col}_3(A)$ form a linearly independent set.

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\text{Col}\, A$.

(b) We have $\text{Col}(A^T A) = \text{Col}\, A^T$, so

$$\text{rank}(A^T A) = \text{rank}(A^T) = \text{rank}(A) = 2.$$

(c) Note that $A^T A$ is a symmetric 3×3 matrix with rank 2.

Therefore it has two nonzero eigenvalues and one eigenvalue equal to zero.

Moreover $A^T A$ is positive semi-definite, so the nonzero eigenvalues are positive.

So the minimum eigenvalue is equal to 0.

(d) Since $\underset{\sim}{v} \in \text{Col}\, A$, the projection yields $\underset{\sim}{v}$.
Since $\underset{\sim}{u} \in (\text{Col}\, A)^\perp$, the projection yields $\underset{\sim}{0}$.
To see that $\underset{\sim}{u} \in (\text{Col}\, A)^\perp$, note that it is orthogonal to every column in A.

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(a) $$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\underset{\sim}{Y} = X \underset{\sim}{b} + \underset{\sim}{e}$$

(b) Let $\underset{\sim}{\hat{\varepsilon}} = \begin{bmatrix} \hat{\varepsilon}_1 \\ \vdots \\ \hat{\varepsilon}_n \end{bmatrix} = (I - P_X)\underset{\sim}{y}$ and $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then

$$\sum_{i=1}^n \hat{\varepsilon}_i x_i = \underset{\sim}{\hat{\varepsilon}}^T \underset{\sim}{x} = \left[(I - P_X)\underset{\sim}{y}\right]^T \underset{\sim}{x} = \underset{\sim}{y}^T \underbrace{(I - P_X)\underset{\sim}{x}}_{=0 \text{ since } \underset{\sim}{x} \in \operatorname{Col} X} = 0.$$

(c) We have

$$\sum_{i=1}^n \hat{\varepsilon}_i = \underset{\sim}{1}_n^T \underset{\sim}{\hat{\varepsilon}} = \underset{\sim}{1}_n^T (I - P_X)\underset{\sim}{y} = \Big[\underbrace{(I - P_X)\underset{\sim}{1}_n}_{=0,\text{ since } \underset{\sim}{1}_n \in \operatorname{Col} X}\Big]^T \underset{\sim}{y} = 0.$$

(d) We have

$$SST = \underset{\sim}{y}^T (I - P_{\underset{\sim}{1}_n})\underset{\sim}{y}, \quad SSE = \underset{\sim}{y}^T (I - P_X)\underset{\sim}{y}, \quad SSR = \underset{\sim}{y}^T (P_X - P_{\underset{\sim}{1}_n})\underset{\sim}{y}.$$

(e) We have

$$\begin{aligned} SST &= \underset{\sim}{y}^T (I - P_{\underset{\sim}{1}_n})\underset{\sim}{y} \\ &= \underset{\sim}{y}^T (I - P_X + P_X - P_{\underset{\sim}{1}_n})\underset{\sim}{y} \\ &= \underset{\sim}{y}^T (I - P_X)\underset{\sim}{y} + \underset{\sim}{y}^T (P_X - P_{\underset{\sim}{1}_n})\underset{\sim}{y} \\ &= SSE + SSR \end{aligned}$$

(f) We have

(i)
$$\begin{aligned} \frac{SSE}{\sigma^2} = \frac{\underset{\sim}{y}^T (I - P_X)\underset{\sim}{y}}{\sigma^2} &\sim \chi^2_{\underbrace{\operatorname{rank}(I - P_X)}_{n-2}}\Big(\phi = \underbrace{\tfrac{1}{\sigma^2}(X\underset{\sim}{b})^T (I - P_X) X\underset{\sim}{b}}_{=0}\Big) \\ &= \chi^2_{n-2} \end{aligned}$$

(ii)
$$\begin{aligned} \frac{SSR}{\sigma^2} = \frac{\underset{\sim}{y}^T (P_X - P_{\underset{\sim}{1}_n})\underset{\sim}{y}}{\sigma^2} &\sim \chi^2_{\underbrace{\operatorname{rank}(P_X - P_{\underset{\sim}{1}_n})}_{1}}\Big(\phi = \tfrac{1}{\sigma^2}(X\underset{\sim}{b})^T (P_X - P_{\underset{\sim}{1}}) X\underset{\sim}{b}\Big) \\ &= \chi^2_1\Big(\phi = \frac{1}{\sigma^2}\sum_{i=1}^n \beta_1^2 (x_i - \bar{x}_n)^2\Big), \end{aligned}$$

since

$$(X\underset{\sim}{b})^T (P_X - P_{\underset{\sim}{1}}) X\underline{b} = \| (P_X - P_1) X\underline{b} \|^2$$

$$= \| X\underset{\sim}{b} - \underset{\sim}{1}_n \frac{1}{n}\sum_{i=1}^n (\beta_0 + \beta_1 x_i) \|^2$$

$$= \sum_{i=1}^n \left(\beta_0 + \beta_1 x_i - (\beta_0 + \beta_1 \bar{x}_n) \right)^2$$

$$= \sum_{i=1}^n \beta_1^2 (x_i - \bar{x}_n)^2 .$$

(g) Use

$$F_{test} = \frac{SSR}{SSE/(n-2)} \sim F_{1,n-2}\left(\phi = \frac{\beta_1^2}{\sigma^2} \sum_{i=1}^n (x_i - \bar{x}_n)^2 \right) .$$

Reject H_0 when $F_{test} > \underbrace{F_{1,n-2,\alpha}}_{\text{upper } \alpha \text{ quantile of central } F_{1,n-2}}$.

(h) If β_1^2 increases, the power increases.

If σ^2 increases, the power decreases

If $\sum_{i=1}^n (x_i - \bar{x}_n)^2$ increases, the power increases.

(i) We have $V = \sigma_\varepsilon^2 I_n$. So

$$\ell_R(\sigma_\varepsilon^2; \underset{\sim}{y}) = -\log|\sigma_\varepsilon^2 I_n| - \log\left| X^T[\sigma_\varepsilon^2 I_n]X \right|$$

$$- \underset{\sim}{y}\left([\sigma_\varepsilon^2 I_n]^{-1} - [\sigma_\varepsilon^2 I_n]^{-1} X (X^T [\sigma_\varepsilon^2 I_n]^{-1} X)^{-1} X^T [\sigma_\varepsilon^2 I_n]^{-1} \right) \underset{\sim}{y}$$

$$= -n \log \sigma_\varepsilon^2 - \log(\sigma_\varepsilon^2)^2 - \log|X^T X|$$

$$- \frac{1}{\sigma_\varepsilon^2} \underset{\sim}{y}^T \left(I - X(X^T X)^{-1} X^T \right) \underset{\sim}{y}$$

$$= \text{const.} - (n-2)\log \sigma_\varepsilon^2 - \frac{1}{\sigma_\varepsilon^2} SSE$$

Now

$$\frac{\partial}{\partial \sigma_\varepsilon^2} \ell_R(\sigma_\varepsilon^2 ; \underset{\sim}{y}) = \frac{-(n-2)}{\sigma_\varepsilon^2} + \frac{SSE}{\sigma_\varepsilon^4} \overset{set}{=} 0$$

$$\Rightarrow \quad \hat{\sigma}^2_{\varepsilon, REML} = \frac{SSE}{n-2} .$$

$\boxed{3}$ (a)

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \vdots \\ \underset{\sim}{y}_a \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x}_1 \\ \vdots \\ \underset{\sim}{x}_a \end{bmatrix} \beta + \begin{bmatrix} \underset{\sim}{x}_1 & & \\ & \ddots & \\ & & \underset{\sim}{x}_a \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_a \end{bmatrix} + \underset{\sim}{e}$$

$$\underset{\sim}{y} = X \underset{\sim}{b} + Z \underset{\sim}{u} + \underset{\sim}{e}$$

(b) We have

$$\begin{aligned} \operatorname{Cov} \underset{\sim}{y} &= Z \operatorname{Cov} \underset{\sim}{u} Z^T + \operatorname{Cov} \underset{\sim}{e} \\ &= \sigma_B^2 Z Z^T + \sigma_\varepsilon^2 I_{na} \\ &= \sigma_B^2 \begin{bmatrix} \underset{\sim}{x}_1 & & \\ & \ddots & \\ & & \underset{\sim}{x}_a \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_1^T & & \\ & \ddots & \\ & & \underset{\sim}{x}_a^T \end{bmatrix} + \sigma_\varepsilon^2 \begin{bmatrix} I_n & & \\ & \ddots & \\ & & I_n \end{bmatrix} \\ &= \begin{bmatrix} \sigma_B^2 \underset{\sim}{x}_1 \underset{\sim}{x}_1^T + \sigma_\varepsilon^2 I_n & & \\ & \ddots & \\ & & \sigma_B^2 \underset{\sim}{x}_a \underset{\sim}{x}_a^T + \sigma_\varepsilon^2 I_n \end{bmatrix} \end{aligned}$$

(c) We obtain

$$V^{-1} = \begin{bmatrix} \frac{1}{\sigma_\varepsilon^2}\left(I_n - \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_1\|^2} \underset{\sim}{x}_1 \underset{\sim}{x}_1^T \right) & & \\ & \ddots & \\ & & \frac{1}{\sigma_\varepsilon^2}\left(I_n - \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_a\|^2} \underset{\sim}{x}_a \underset{\sim}{x}_a^T \right) \end{bmatrix}$$

(d) Using $\hat{\beta}_{gls} = (X^T V^{-1} X)^{-1} X^T V^{-1} \underset{\sim}{y}$, we write

$$X^T V^{-1} X = \begin{pmatrix} \underset{\sim}{x}_1^T & & \\ & \ddots & \\ & & \underset{\sim}{x}_n^T \end{pmatrix} V^{-1} \begin{pmatrix} \underset{\sim}{x}_1 & & \\ & \ddots & \\ & & \underset{\sim}{x}_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \underset{\sim}{x}_i^T \left(\frac{1}{\sigma_\varepsilon^2} \left(I_n - \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i \underset{\sim}{x}_i^T \right) \right) \underset{\sim}{x}_i$$

$$= \sum_{i=1}^{n} \frac{1}{\sigma_\varepsilon^2} \left(\|\underset{\sim}{x}_i\|^2 - \frac{\|\underset{\sim}{x}_i\|^2 \sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \|\underset{\sim}{x}_i\|^2 \right)$$

$$= \sum_{i=1}^{n} \|\underset{\sim}{x}_i\|^2 \frac{1}{\sigma_\varepsilon^2} \left(1 - \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \right)$$

$$= \sum_{i=1}^{n} \frac{\|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2}$$

and

$$X^T V^{-1} \underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{x}_1^T & & \\ & \ddots & \\ & & \underset{\sim}{x}_n^T \end{bmatrix} V^{-1} \begin{bmatrix} \underset{\sim}{y}_1 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \underset{\sim}{x}_i^T \left(\frac{1}{\sigma_\varepsilon^2} \left(I_n - \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i \underset{\sim}{x}_i^T \right) \right) \underset{\sim}{y}_i$$

$$= \sum_{i=1}^{n} \frac{1}{\sigma_\varepsilon^2} \left(\underset{\sim}{x}_i^T \underset{\sim}{y}_i - \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i^T \underset{\sim}{y}_i \right)$$

$$= \sum_{i=1}^{n} \frac{\underset{\sim}{x}_i^T \underset{\sim}{y}_i}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2}$$

So we obtain

$$\hat{\beta}_{gls} = \frac{\sum_{i=1}^{n} x_i^T \underset{\sim}{y}_i \,/\, (\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2)}{\sum_{i=1}^{n} \|\underset{\sim}{x}_i\|^2 \,/\, (\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2)}$$

(e) Write

$$\tilde{v}_i = \hat{\beta}_{gls} + \underset{\sim}{e}_i^T \underbrace{\left[\sigma_B^2 I_n\right]}_{G} \underbrace{\begin{bmatrix} \underset{\sim}{x}_1^T & & \\ & \ddots & \\ & & \underset{\sim}{x}_n^T \end{bmatrix}}_{Z^T} V^{-1} \begin{bmatrix} \underset{\sim}{y}_1 - \underset{\sim}{x}_1 \hat{\beta}_{gls} \\ \vdots \\ \underset{\sim}{y}_n - \underset{\sim}{x}_n \hat{\beta}_{gls} \end{bmatrix}$$

i^{th} elem basis vector

$$= \hat{\beta}_{gls} + \sigma_B^2 \; \underset{\sim}{x}_i^T \left(\frac{1}{\sigma_\varepsilon^2} \left(I_n - \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i \underset{\sim}{x}_i^T \right) \right) \left(\underset{\sim}{y}_i - \underset{\sim}{x}_i \hat{\beta}_{gls} \right)$$

$$= \hat{\beta}_{gls} + \sigma_B^2 \left[\frac{1}{\sigma_\varepsilon^2} \left(\underset{\sim}{x}_i^T (\underset{\sim}{y}_i - \underset{\sim}{x}_i \hat{\beta}_{gls}) - \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i^T (\underset{\sim}{y}_i - \underset{\sim}{x}_i \hat{\beta}_{gls}) \right) \right]$$

$$= \hat{\beta}_{gls} + \frac{\sigma_B^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \underset{\sim}{x}_i^T \left(\underset{\sim}{y}_i - \underset{\sim}{x}_i \hat{\beta}_{gls} \right)$$

$$= \hat{\beta}_{gls} + \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \frac{\underset{\sim}{x}_i^T \underset{\sim}{y}}{\|\underset{\sim}{x}_i\|^2} - \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \hat{\beta}_{gls}$$

$$= \left(\frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \right) \frac{\underset{\sim}{x}_i^T \underset{\sim}{y}}{\|\underset{\sim}{x}_i\|^2} + \left(1 - \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} \right) \hat{\beta}_{gls} ,$$

so

$$\tau_i = \frac{\sigma_B^2 \|\underset{\sim}{x}_i\|^2}{\sigma_\varepsilon^2 + \sigma_B^2 \|\underset{\sim}{x}_i\|^2} .$$