STAT 214 HW 01 SOLUTIONS

$\boxed{2}$ Four treatments will be compared in an experiment in which four subjects are assigned to each treatment according to a Latin Square design.

There are two blocking variables – row and column blocking variables – with four levels each.

The block and treatment arrangement will follow the diagram

	B_1	B_2	B_3	B_4
A_1	1	2	3	4
A_2	2	1	4	3
A_3	3	4	2	1
A_4	4	3	1	2

,

where A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are block effects, and the numbers in the cell indicate what treatment is applied at the block combinations.

The resulting data will be analyzed assuming the linear model

$$y_{ijk} = \mu + A_i + B_j + \alpha_k + \varepsilon_{ijk} \quad , \quad i, j, k = 1,2,3,4 \, ,$$

where

- μ is a mean
- $\alpha_1, \ldots, \alpha_4$ are treatment effects
- A_i are indep. $N(0, \sigma_A^2)$
- B_i are indep. $N(0, \sigma_B^2)$
- ε_{ijk} are indep. $N(0, \sigma_\varepsilon^2)$
- A_i, B_i, and ε_{ijk} are independent.

Write the linear model in matrix notation

$$\underset{\sim}{y} = X \underset{\sim}{b} + Z \underset{\sim}{u} + \underset{\sim}{\varepsilon} \, ,$$

where $\underset{\sim}{b}$ contains fixed parameters and $\underset{\sim}{u}$ contains random effects.

Write out the entries of each vector and matrix.

Solution:

$$
\begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{14} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{31} \\ Y_{32} \\ Y_{33} \\ Y_{34} \\ Y_{41} \\ Y_{42} \\ Y_{43} \\ Y_{44} \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & & & \\
1 & & 1 & & \\
1 & & & 1 & \\
1 & & & & 1 \\
1 & & 1 & & \\
1 & 1 & & & \\
1 & & & & 1 \\
1 & & & 1 & \\
1 & & & 1 & \\
1 & & & & 1 \\
1 & & 1 & & \\
1 & 1 & & & \\
1 & & & & 1 \\
1 & & & 1 & \\
1 & 1 & & & \\
1 & & 1 & &
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}
+
\begin{bmatrix}
1 & & & & 1 & & & \\
1 & & & & & 1 & & \\
1 & & & & & & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & 1 & & & \\
 & 1 & & & & 1 & & \\
 & 1 & & & & & 1 & \\
 & 1 & & & & & & 1 \\
 & & 1 & & 1 & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & 1 & \\
 & & 1 & & & & & 1 \\
 & & & 1 & 1 & & & \\
 & & & 1 & & 1 & & \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & 1
\end{bmatrix}
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}
+
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{14} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{24} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \\ \varepsilon_{34} \\ \varepsilon_{41} \\ \varepsilon_{42} \\ \varepsilon_{43} \\ \varepsilon_{44} \end{bmatrix}
$$

$\boxed{2}$ For a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D invertible, verify that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BE^{-1}CA^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}CA^{-1} & E^{-1} \end{bmatrix},$$

where $E = D - CA^{-1}B$

Solution:

$$\begin{bmatrix} A^{-1} + A^{-1}BE^{-1}CA^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}CA^{-1} & E^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$= \begin{bmatrix} I + A^{-1}BE^{-1}C - A^{-1}BE^{-1} & A^{-1}B + A^{-1}BE^{-1}CA^{-1}B - A^{-1}BE^{-1}D \\ -E^{-1}C + E^{-1}C & -E^{-1}CA^{-1}B + E^{-1}D \end{bmatrix}$$

Use $D = E + CA^{-1}B$

$$= \begin{bmatrix} I & A^{-1}B + A^{-1}BE^{-1}CA^{-1}B - A^{-1}BE^{-1}(E + CA^{-1}B) \\ 0 & -E^{-1}CA^{-1}B + E^{-1}(E + CA^{-1}B) \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

$\boxed{3}$ Let X be an $n \times p$ matrix.

Describe the change in X when it is premultiplied by $\left(I_n - \frac{1}{n}\underset{\sim}{1}_n \underset{\sim}{1}_n^T\right)$.

Solution: We have

$$\left(I_n - \frac{1}{n}\underset{\sim}{1}_n \underset{\sim}{1}_n^T\right) X = \left(I_n - \frac{1}{n}\underset{\sim}{1}_n \underset{\sim}{1}_n^T\right)\left[\underset{\sim}{x}_1 \cdots \underset{\sim}{x}_p\right]$$

$$= \left[\underset{\sim}{x}_1 - \left(\frac{1}{n}\sum_{i=1}^{n} x_{1i}\right)\underset{\sim}{1}_n \cdots \underset{\sim}{x}_p - \left(\frac{1}{n}\sum_{i=1}^{n} x_{pi}\right)\underset{\sim}{1}_n\right],$$

So premultiplication of X by $\left(I_n - \frac{1}{n}\underset{\sim}{1}_n \underset{\sim}{1}_n^T\right)$ centers the columns of X so that they have mean zero.

4 Characterize the solution set of $A\underset{\sim}{x} = \underset{\sim}{b}$ (provided the system of equations is consistent), where

$$A = \begin{bmatrix} 3 & 5 & -2 \\ -3 & -2 & -1 \\ 6 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}.$$

Solution: Row-reduce the augmented matrix:

$$[A \;\; b] = \begin{bmatrix} 3 & 5 & -2 & -7 \\ -3 & -2 & -1 & 1 \\ 6 & 1 & 5 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -2 & -7 \\ 0 & 3 & -3 & -6 \\ 0 & -9 & 9 & 18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -2 & -7 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -2 & -7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 3 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 + x_3 = 1 \\ x_2 - x_3 = -2 \\ x_3 \text{ free} \end{array} \iff \begin{array}{l} x_1 = 1 - x_3 \\ x_2 = -2 + x_3 \\ x_3 = x_3 \end{array}$$

So the solution set is given by

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \; x_3 \in \mathbb{R} \right\}$$

5 Show that if two nonzero vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are orthogonal, then $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ is linearly independent.

<u>**Solution:**</u> Orthogonality of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ means $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 0$.

Suppose $\underset{\sim}{v}_1 c_1 + \underset{\sim}{v}_2 c_2 = 0$ for c_1 and c_2 not both zero.

If $c_1 = 0$ but $c_2 \neq 0$, then $\underset{\sim}{v}_2$ must be the zero vector. But $\underset{\sim}{v}_2$ is nonzero.

If $c_2 = 0$ but $c_1 \neq 0$, then $\underset{\sim}{v}_1$ must be the zero vector. But $\underset{\sim}{v}_1$ is nonzero.

If $c_1 \neq 0$ and $c_2 \neq 0$, then

$$\underset{\sim}{v}_1 c_1 = -\underset{\sim}{v}_2 c_2 \quad \Rightarrow \quad \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 c_1 = -\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 c_2$$

$$\Rightarrow \quad 0 = -\|\underset{\sim}{v}_2\| c_2 \neq 0,$$

so we have a contradiction.

Therefore $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are linearly independent.

$\boxed{6}$ Let $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ be a set of linearly independent vectors in $\mathbb{R}^n$ and let

$$\underset{\sim}{u}_1 = \underset{\sim}{v}_1 \qquad \text{and} \qquad \underset{\sim}{u}_2 = \underset{\sim}{v}_2 - \left(\frac{\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1}\right)\underset{\sim}{v}_1 .$$

Show that $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}$ is linearly independent. Hint: Show that $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are orthogonal.

<u>Solution</u>: We have

$$\begin{aligned}
\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 &= \underset{\sim}{v}_1 \cdot \left[\underset{\sim}{v}_2 - \left(\frac{\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1}\right)\underset{\sim}{v}_1\right] \\
&= \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 \left(\frac{\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1}{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1}\right) \\
&= 0.
\end{aligned}$$

Since $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are orthogonal, $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2\}$ is linearly independent.