$\boxed{1}$ Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p\}$ and let $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_q\}$ be an orthogonal basis for $W^\perp$.

(a) Show that the set $\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_q\}$ is linearly independent.

(b) State whether $\text{Span}\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_q\} = \mathbb{R}^n$. Give your reasoning.

(c) Show that $p + q = n$.

(d) Show whether the statement is true or not true: For every vector $\underset{\sim}{x} \in \mathbb{R}^n$ we have $x \in W$ or $x \in W^\perp$.

<u>Solution</u>:

(a) Write

$$c_1 \underset{\sim}{w}_1 + \dots + c_p \underset{\sim}{w}_p + d_1 \underset{\sim}{v}_1 + \dots + d_q \underset{\sim}{v}_q = \underset{\sim}{0}. \qquad (*)$$

Check whether this implies $c_1 = \dots = c_p = d_1 = \dots = d_q = 0$.

If so, $\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_q\}$ is linearly independent.

Take the inner product of the vector in $(*)$ with any $\underset{\sim}{v}_k$. This gives

$$\underset{\sim}{v}_k \cdot \left(c_1 \underset{\sim}{w}_1 + \dots + c_p \underset{\sim}{w}_p + d_1 \underset{\sim}{v}_1 + \dots + d_q \underset{\sim}{v}_q\right) = d_k \underset{\sim}{v}_k \cdot \underset{\sim}{v}_k = 0$$

So we must have $d_k = 0$. Do this for each $k = 1, \dots, q$.

Likewise, take

$$\underset{\sim}{w}_j \cdot \left(c_1 \underset{\sim}{w}_1 + \dots + c_p \underset{\sim}{w}_p + d_1 \underset{\sim}{v}_1 + \dots + d_q \underset{\sim}{v}_q\right) = c_j \underset{\sim}{w}_j \cdot \underset{\sim}{w}_j = 0,$$

so we must have $c_j = 0$. Do this for each $j = 1, \dots, p$.

So $(*)$ implies $c_1 = \dots = c_p = d_1 = \dots = d_q = 0$.

Therefore $\{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_p, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_q\}$ is a linearly independent set.

OR: Just say $\mathbf{w}_j \cdot \mathbf{w}_k = 0 \;\forall j \neq k$, $\mathbf{w}_j \cdot \mathbf{v}_k = 0 \;\forall j, k$, $\mathbf{v}_j \cdot \mathbf{v}_k = 0 \;\forall j \neq k$.

Therefore $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

Since orthogonality of a set implies linear dependence, this suffices.

(b) By the orthogonal decomposition theorem, any vector $\mathbf{y} \in \mathbb{R}^n$ can be written

$$\mathbf{y} = \hat{\mathbf{y}} + \hat{\mathbf{e}}, \qquad \hat{\mathbf{y}} \in W \text{ and } \hat{\mathbf{e}} \in W^\perp.$$

Since $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ is a basis for W and $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ is a basis for $W^\perp$,

$$\hat{\mathbf{y}} = c_1 \mathbf{w}_1 + \dots + c_p \mathbf{w}_p \quad \text{for some } c_1, \dots, c_p \in \mathbb{R}$$

and $$\hat{\mathbf{e}} = d_1 \mathbf{v}_1 + \dots + d_q \mathbf{v}_q \quad \text{for some } d_1, \dots, d_q \in \mathbb{R}.$$

So

$$\mathbf{y} = c_1 \mathbf{w}_1 + \dots + c_p \mathbf{w}_p + d_1 \mathbf{v}_1 + \dots + d_q \mathbf{v}_q.$$

Since $\mathbf{y}$ is any vector in $\mathbb{R}^n$, we have $\text{Span}\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\} = \mathbb{R}^n$.

(c) Since $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is a basis for $\mathbb{R}^n$, which we know because

$\text{Span}\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\} = \mathbb{R}^n$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is linearly indep.,

the number of vectors in $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is equal to the dimension of $\mathbb{R}^n$.

Since $\dim \mathbb{R}^n = n$, we have $p + q = n$.

(d) The statement is false. We can give a counter example:

The set $W = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^2$ with orthogonal complement $W^\perp = \text{Span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

The vector $\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}$ is in $\mathbb{R}^2$ but $\mathbf{x} \notin W$ and $\mathbf{x} \notin W^\perp$.

So orthogonal complements do not partition the vector space; rather they allow for each $\mathbf{x}$ the unique representation $\mathbf{x} = \hat{\mathbf{x}} + \hat{\mathbf{e}}$, where $\hat{\mathbf{x}} \in W$ and $\hat{\mathbf{e}} \in W^\perp$.

$\boxed{2}$ Let $U = [\mathbf{u}_1 \cdots \mathbf{u}_p]$, where $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$. Show that the orthogonal projection of $\mathbf{y} \in \mathbb{R}^n$ onto W is given by

$$\hat{\mathbf{y}} = U U^T \mathbf{y}.$$

<u>Solution</u>: We have

$$\begin{aligned}
\hat{\mathbf{y}} &= U U^T \mathbf{y} \\
&= [\mathbf{u}_1 \cdots \mathbf{u}_p] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_p^T \end{bmatrix} \mathbf{y} \\
&= \left(\mathbf{u}_1^T \mathbf{y}\right) \mathbf{u}_1 + \cdots + \left(\mathbf{u}_p^T \mathbf{y}\right) \mathbf{u}_1 \\
&= \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\right) \mathbf{u}_p,
\end{aligned}$$

which is the orthogonal projection $\mathrm{proj}_W \mathbf{y}$ of $\mathbf{y}$ onto W.

$\boxed{3}$ Let $\underset{\sim}{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and let $\underset{\sim}{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $\underset{\sim}{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

(a) Produce an orthonormal basis for Span $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$.

(b) Give the orthogonal projection $\hat{\underset{\sim}{y}}$ of $\underset{\sim}{y}$ onto Span $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$.

Solution:

(a) First use Gram-Schmidt to orthogonalize the basis: let

$$\underset{\sim}{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{u}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \left(\frac{\begin{bmatrix} 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - \frac{1}{2} 2 \\ -1 - \frac{1}{2}(-5) \\ 2 - \frac{1}{2} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \leftarrow \text{can multiply by } \frac{2}{3}$$

An orthogonal basis is $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

To obtain an orthonormal basis, normalize each vector: We have

$$\left\| \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\| = \sqrt{30} \quad , \quad \left\| \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{6} .$$

So an orthonormal basis is

$$\left\{ \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} .$$

(b) The orthogonal projection of $\underset{\sim}{y}$ onto Span $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ is given by

$$\hat{\underset{\sim}{y}} = \left(\frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} + \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{30} (-2) \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} + \frac{1}{6} (4) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{30} \left(\begin{bmatrix} -4 \\ 10 \\ -2 \end{bmatrix} + 20 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= \frac{1}{30} \begin{bmatrix} 36 \\ 30 \\ -18 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 12 \\ 10 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 6/5 \\ 1 \\ -3/5 \end{bmatrix} .$$

4 Show that if W and V are subspaces of $\mathbb{R}^n$ such that $W \subset V$, then

$$\dim W \leq \dim V.$$

Solution:

Suppose V has dimension p.

Suppose $\{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_q\}$ is a basis for W with $q > p$ (which would make $\dim W > \dim V$).

Since $\{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_q\}$ is a basis for W, and $W \subset V$, we have $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_q \in V$.

But since V has dimension p, any set of more than p vectors in V must be linearly dependent, by the Dimension Theorem.

Therefore $\{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_q\}$ cannot be a basis, which is a contradiction.

This implies we cannot have $\dim W > \dim V$, so $\dim W \leq \dim V$.

$\boxed{5}$ Let $A = \sum_{k=1}^{r} \underset{\sim}{u}_k \underset{\sim}{v}_k^T$ for some vectors $\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r \in \mathbb{R}^m$ and $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_r \in \mathbb{R}^n$.

Show that rank $A \leq r$.

Solution 1:

Let $\underset{\sim}{y} \in \text{Col}\, A$. Then there is some $\underset{\sim}{x} \in \mathbb{R}^n$ such that

$$\underset{\sim}{y} = A\underset{\sim}{x} = \sum_{k=1}^{r} \underset{\sim}{u}_k \underbrace{\underset{\sim}{v}_k^T \underset{\sim}{x}}_{\text{coefficients on } \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r} \in \text{Span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}.$$

Therefore $\text{Col}\, A \subset \text{Span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$.

The dimension of $\text{Span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$ is at most r.

This is so because if $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$ is linearly dependent, it is a basis for $\text{Span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$.
If $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$ is not linearly independent, we must remove one or more vectors to obtain a basis for $\text{Span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_r\}$

Therefore $\dim \text{Col}\, A \leq r$.

Solution 2:

Let $U = [\underset{\sim}{u}_1 \cdots \underset{\sim}{u}_r]$ and $V = [\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_r]$.

Then $A = \sum_{k=1}^{r} \underset{\sim}{u}_k \underset{\sim}{v}_k^T = UV^T$.

Then
$$\text{rank}\, A \leq \min\{\text{rank}\, U, \text{rank}\, V^T\}$$
$$= \min\{\text{rank}\, U, \text{rank}\, V\}.$$

Since U and V have r columns, their ranks cannot exceed r.

So we have rank $A \leq r$.

$\boxed{6}$ Consider the linear model

$$Y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}, \qquad i = 1, 2, \quad j = 1, 2, 3,$$

where $x_{ij} = j$ for $i = 1, 2$, $j = 1, 2, 3$, and the ε_{ij} are $N(0, \sigma^2)$.

(a) Put the model equations into matrix form $\underset{\sim}{Y} = X \underset{\sim}{b} + \underset{\sim}{\varepsilon}$.

(b) Give a basis for Col X.

(c) Give rank X.

(d) Give dim Nul X.

(e) Give $\dim (\text{Col } X)^{\perp}$.

(f) Give a basis for the orthogonal complement of Col X.

(g) Give the orthogonal projection of

$$\underset{\sim}{Y} = (5, 6, 8, 4, 3, 1)^T$$

onto Nul X^T.

(h) Give the orthogonal projection of the same $\underset{\sim}{Y}$ vector onto Col X.

Solution:

(a)
$$\underbrace{\begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{bmatrix}}_{\underset{\sim}{Y}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 3 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\underset{\sim}{b}} + \underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{bmatrix}}_{\underset{\sim}{\varepsilon}}$$

(b) By looking at X, we see that the columns are linearly dependent. The first column is the sum of the second and third columns, so by the Spanning Set Theorem, it can be removed, and the remaining columns will still span Col X.

Columns 2 through 4 of X form a linearly independent set and span Col X. Therefore they form a basis for Col X.

Therefore the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is a basis for Col X.

(c) We have $\text{rank } X = \dim \text{Col } X$, and the dimension of a subspace is the number of vectors in a basis. So we have

$$\text{rank } X = 4.$$

(d) The dimension of Nul X must be 1, since $\text{rank } X = 4$ and

$$\text{rank } X + \dim \text{Nul } X = \#\text{ columns of } X = 5.$$

(e) We have $(\text{Col } X)^{\perp} = \text{Nul } X^T$. The dimension of this satisfies

$$\dim \text{Col } X^T + \dim \text{Nul } X^T = \#\text{ rows of } X = 6.$$

Since $\dim \text{Col } X^T = \text{rank } X^T = \text{rank } X = 4$, by the Rank Theorem,

$$\dim \text{Nul } X^T = 2.$$

(f) We have $(\mathrm{Col}\, X)^{\perp} = \mathrm{Nul}\, X^T$. To find $\mathrm{Nul}\, X^T$, we row-reduce X^T.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix} \sim$$

$\dim \mathrm{Col}\, X = 4$

$(\mathrm{Col}\, X)^{\perp} = \mathrm{Nul}\, X^T$

$\dim (\mathrm{Col}\, X)^{\perp} = 2$

$\dim \mathrm{Col}\, X^T + \dim \mathrm{Nul}\, X^T = 6$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_4 - x_6 &= 0 \\ x_5 + 2x_6 &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

So a basis for $(\operatorname{Col} X)^{\perp} = \operatorname{Nul} X^T$ is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(g) The orthogonal projection of $\underset{\sim}{y}$ onto $\operatorname{Nul} X^T$ is given by

$$\underset{\sim}{\hat{e}} = \frac{\begin{pmatrix} 5 \\ 6 \\ 8 \\ 4 \\ 3 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 5 \\ 6 \\ 7 \\ 4 \\ 3 \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left(-\frac{1}{6}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -2/6 \\ 1/6 \\ -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

(h) The orthogonal projection of $\underset{\sim}{y}$ onto $\operatorname{Col} X$ is equal to

$$\underset{\sim}{\hat{y}} = \underset{\sim}{y} - \underset{\sim}{\hat{e}},$$

by the orthogonal decomposition theorem, since $\operatorname{Col} X$ and $\operatorname{Nul} X^T$ are orthogonal complements.

We have

$$\underset{\sim}{\hat{y}} = \begin{bmatrix} 5 \\ 6 \\ 8 \\ 4 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -2/6 \\ 1/6 \\ -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}.$$