STAT 714 hw 3

Eigenvalues, eigenvectors, trace, determinant, quadratic forms, symmetric matrices

1. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Show that if $\mathbf{A}$ is not invertible, then $\mathbf{AB}$ is not invertible.

2. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times m$ matrix.
   (a) Show that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.
   (b) Show that $\mathrm{tr}(\mathbf{A}^T\mathbf{A}) = \|\mathbf{A}\|_F^2$, where $\|\cdot\|_F$ is the *Frobenius norm*, which is defined as the square root of the sum of the squared entries of a matrix.

3. Show that for any $m \times n$ matrix $\mathbf{A}$, the matrix $\mathbf{A}^T\mathbf{A}$ is positive semidefinite.

4. For a 3×3 matrix $\mathbf{A}$, show that the characteristic polynomial $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$ is given by

$$p_{\mathbf{A}}(t) = t^3 - (\mathrm{tr}\,\mathbf{A})t^2 + \cdots + (-1)^3 \det \mathbf{A}.$$

5. Show that if $\mathbf{U}$ is an orthogonal matrix, then $\det \mathbf{U}$ is equal to 1 or -1.

6. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
   (a) Give $\mathrm{rank}\,\mathbf{A}$.
   (b) Give the number of nonzero eigenvalues of $\mathbf{A}$.
   (c) Give $\mathrm{rank}\,\mathbf{A}^T\mathbf{A}$.
   (d) Give the number of nonzero eigenvalues of $\mathbf{A}^T\mathbf{A}$.
   (e) Argue whether the statement is true or not, and whether or not it is true under some conditions: "The rank of a matrix is equal to its number of nonzero eigenvalues."

7. A matrix $\mathbf{A}$ is called *idempotent* if $\mathbf{AA} = \mathbf{A}$. Show that for any idempotent matrix:
   (a) The eigenvalues are 0 or 1.
   (b) The determinant is 0 or 1.
   (c) If the determinant is 1, the matrix is the identity matrix.
   (d) If $\mathbf{A}$ is symmetric and idempotent, show that $\mathrm{rank}\,\mathbf{A} = \mathrm{tr}\,\mathbf{A}$.

8. Let $\mathbf{A}$ be a symmetric matrix. Show that the values

$$m = \inf_{\|\mathbf{x}\|=1} \{\mathbf{x}^T\mathbf{A}\mathbf{x}\} \quad \text{and} \quad M = \sup_{\|\mathbf{x}\|=1} \{\mathbf{x}^T\mathbf{A}\mathbf{x}\}$$

   are equal to the least and greatest eigenvalues of $\mathbf{A}$, respectively.

9. Let the random variable triplet (X_1, X_2, X_3) have covariance matrix $\mathbf{\Sigma} = (1-\tau)\mathbf{I}_3 + \tau\mathbf{1}_3\mathbf{1}_3^T$.
   (a) Show that $1-\tau$ and $1+2\tau$ are eigenvalues of $\mathbf{\Sigma}$.
   (b) Give the range of values for τ for which $\mathbf{\Sigma}$ is positive definite.
   (c) Suppose $\tau > 0$. Give values v_1, v_2, and v_3 such that $v_1X_1 + v_2X_2 + v_3X_3$ has the smallest possible variance while $v_1^2 + v_2^2 + v_3^2 = 1$.