$\boxed{1}$ Let $A$ and $B$ be square matrices.
Show that if $A$ is not invertible then $AB$ is not invertible.

Solution: It is sufficient to show that $AB$ invertible $\Rightarrow$ $A$ invertible.
[By contrapositive].

Suppose $AB$ is invertible. Then there exists a matrix $C$ such that

$$ABC = I$$

Then $BC$ is the inverse of $A$, so $A$ is invertible.

$\boxed{2}$ Let $A$ be $m \times n$ and let $B$ be $n \times m$. Show

(a) $tr(AB) = tr(BA)$.

(b) $tr(A^T A) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2$

Solutions:

(a) We have

$$\begin{aligned} tr(AB) &= \sum_{k=1}^{m} (AB)_{kk} \\ &= \sum_{k=1}^{m} row_k(A)\, col_k(B) \\ &= \sum_{k=1}^{m} \sum_{i=1}^{n} A_{ki} B_{ik} \end{aligned}$$

and

$$\begin{aligned} tr(BA) &= \sum_{i=1}^{n} (BA)_{ii} \\ &= \sum_{i=1}^{n} row_i(B)\, col_i(A) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} B_{ik} A_{ki} \quad . \end{aligned}$$

We see that these are equal.

(b) We have

$$\begin{aligned}
tr\left(A^T A\right) &= \sum_{i=1}^{n} \left(A^T A\right)_{ii} \\
&= \sum_{i=1}^{n} row_i\left(A^T\right) col_i\left(A\right) \\
&= \sum_{i=1}^{n} col_i\left(A\right) col_i\left(A\right) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{m} A_{ki} A_{ki} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 .
\end{aligned}$$

$\boxed{3}$ Show that for any matrix $A$, the matrix $A^T A$ is positive semidefinite.

We must show that $A^T A$ has no negative eigenvalues.

Let $\underset{\sim}{v}$ be an eigenvector of $A^T A$.

Then, for some $\lambda$, we have

$$\begin{aligned}
& A^T A \underset{\sim}{v} = \lambda \underset{\sim}{v} \\
\iff \quad & \underset{\sim}{v}^T A^T A \underset{\sim}{v} = \lambda \underset{\sim}{v}^T \underset{\sim}{v} \\
\iff \quad & \| A \underset{\sim}{v} \|^2 = \lambda \| \underset{\sim}{v} \|^2
\end{aligned}$$

Since $\|A\underset{\sim}{v}\|^2$ and $\|\underset{\sim}{v}\|^2$ are both positive, we have $\lambda \geq 0$.

Therefore $A^T A$ does not have any negative eigenvalues.

In consequence $A^T A$ is positive semi-definite.

$\boxed{4}$ Let $p_A(t) = \det(tI - A)$. Then for a $3\times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we have

$$\begin{vmatrix} t-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & t-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & t-a_{33} \end{vmatrix} = (t-a_{11})\begin{vmatrix} t-a_{22} & -a_{23} \\ -a_{32} & t-a_{33} \end{vmatrix} + a_{21}\begin{vmatrix} -a_{12} & -a_{13} \\ -a_{32} & t-a_{33} \end{vmatrix} - a_{31}\begin{vmatrix} -a_{12} & -a_{13} \\ t-a_{22} & -a_{23} \end{vmatrix}$$

$$= (t-a_{11})\left[(t-a_{22})(t-a_{33}) - a_{23}a_{32}\right]$$

$$+ a_{21}\left[-(t-a_{33})a_{12} - a_{13}a_{32}\right] - a_{31}\left[a_{12}a_{23} + a_{13}(t-a_{22})\right]$$

$$= (t-a_{11})\left[t^2 - (a_{22}+a_{33})t + a_{22}a_{33} - a_{23}a_{32}\right]$$

$$- (t-a_{33})a_{12}a_{21} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} - a_{31}a_{13}(t-a_{22})$$

$$= t^3 - (a_{22}+a_{33})t^2 + a_{22}a_{33}t - a_{23}a_{32}t$$

$$- a_{11}t^2 + a_{11}(a_{22}+a_{33})t - a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32}$$

$$- a_{12}a_{21}t + a_{33}a_{12}a_{21} - a_{21}a_{13}a_{32}$$

$$- a_{31}a_{12}a_{23} - a_{31}a_{13}t + a_{31}a_{13}a_{22}$$

$$= t^3 - (a_{11}+a_{22}+a_{33})t^2$$

$$+ (a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} + a_{11}a_{22} + a_{11}a_{33} - a_{31}a_{13})t$$

$$- a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= t^3 - (\operatorname{tr} A)t^2 + \left( \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \right) t$$

$$+ (-1)^3 (\det A).$$

$\boxed{5}$ For an orthogonal matrix $U$, show that $\det U \in \{-1, 1\}$.

Since $U$ is orthogonal, we have $U^T U = I$, so

$$\det\left(U^T U\right) = \det I = 1.$$

Moreover $\det(U^T U) = (\det U^T)(\det U) = (\det U)(\det U)$.

So we have

$$(\det U)(\det U) = 1,$$

which can only be true if $\det U$ is equal to $-1$ or $1$.

$\boxed{6}$ Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) We have $\operatorname{rank} A = 2$.

(b) We have $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 0) = -\lambda^3$.

Setting this equal to zero gives $\lambda = 0$.

There are <u>zero</u> nonzero eigenvalues.

(c) We have $A^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which has rank 2.

(d) The matrix $A^T A$ has eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, 1)$, so it has <u>two</u> nonzero eigenvalues.

(e) The statement is true for symmetric matrices, but not in general as parts (a) and (b) showed.

It is true for symmetric matrices because of the statement in the Spectral Theorem: "For each eigenvalue, the dimension of the corresponding eigenspace is equal to the multiplicity of the eigenvalue as a root of the characteristic polynomial."

Recall that for a matrix with $n$ columns, the Rank Theorem gives

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Now, $\operatorname{Nul} A = \{ \underset{\sim}{x} : A\underset{\sim}{x} = 0 \}$ is the union of the eigenspaces corresponding to the zero eigenvalues, across multiples.

So the number of eigenvalues equal to zero gives $\dim \operatorname{Nul} A$.

Since symmetric $n \times n$ matrices (by the Spectral Theorem) have $n$ real eigenvalues, we have

$$\operatorname{rank} A = n - \dim \operatorname{Nul} A = n - \#\{\text{eigenvalues equal to } 0\} = \#\{\text{nonzero eigenvalues}\}.$$

$\boxed{7}$ Let $A$ be an idempotent matrix, so $AA = A$. Show:

(a) The eigenvalues of $A$ are equal to 0 or 1.

(b) The determinant of $A$ is equal to 0 or 1

(c) If the determinant is 1 then $A = I$

(d) If $A$ symmetric and idempotent, then rank $A$ = tr $A$.

Solutions:

(a) Let $\lambda$ be a scalar and $\underset{\sim}{x}$ be a nonzero vector such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}.$$

Then multiplying both sides of $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ by $A$ gives

$$A\underset{\sim}{x} = \lambda A \underset{\sim}{x} \Rightarrow \lambda = \lambda^2.$$

So we must have $\lambda$ equal to zero or one.

(b) Since the determinant is the product of the eigenvalues, and these can be only 0 or 1, the determinant can be only 0 or 1.

(c) If $\det A = 1$, then all the eigenvalues must be equal to 1. Since the number 0 is not an eigenvalue of $A$, $A$ is invertible. Combining the facts that $A$ is both idempotent and invertible, we have

$$A^{-1}A = I \iff A^{-1}\underbrace{AA}_{A} = A \iff A^{-1}A = A \iff I = A.$$

(d) If $A$ is symmetric and idempotent, then the multiplicity of 0 as an eigenvalue is equal to the dimension of the null space of $A$.

From statement ② of the Spectral Theorem we have

$$\#\{\text{eigenvalues equal to } 0\} = \dim \operatorname{Nul} A.$$

Since $\operatorname{rank} A + \dim \operatorname{Nul} A = n =$ total number of eigenvalues, we have

$$\operatorname{rank} A = \#\{\text{nonzero eigenvalues}\}.$$

Since all the nonzero eigenvalues are equal to 1, and tr $A$ is equal to the sum of the eigenvalues, we have $\operatorname{tr} A = \#\{\text{nonzero eigenvalues}\}$, i.e.

$$\operatorname{rank} A = \operatorname{tr} A.$$

8. Let $A$ be a symmetric matrix. Show that the quantities

$$m = \inf_{\|\underset{\sim}{x}\|=1} \{\underset{\sim}{x}^T A \underset{\sim}{x}\} \quad \text{and} \quad M = \sup_{\|\underset{\sim}{x}\|=1} \{\underset{\sim}{x}^T A \underset{\sim}{x}\}$$

are equal to the least and the greatest eigenvalues of $A$, respectively.

Solution:

Let $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ be an orthogonal set of unit eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$.

Then we may write the spectral decomposition of $A$ as

$$A = \sum_{k=1}^{n} \lambda_k \underset{\sim}{u}_k \underset{\sim}{u}_k^T$$

Note that since $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ are orthogonal, they from a basis for $\mathbb{R}^n$.

So for any vector $\underset{\sim}{x} \in \mathbb{R}^n$, we may write $\underset{\sim}{x} = c_1 \underset{\sim}{u}_1 + \cdots + c_n \underset{\sim}{u}_n$ for some $c_1, \ldots, c_n$.

Then we have

$$\left(\frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}^T\right) A \left(\frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}\right) = \frac{\sum_{k=1}^{n} \lambda_k \left(\underset{\sim}{x}^T \underset{\sim}{u}_k\right)^2}{\underset{\sim}{x}^T \underset{\sim}{x}}$$

$$= \frac{\sum_{k=1}^{n} \lambda_k \left( (c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p)^T \underset{\sim}{u}_k \right)^2}{(c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p)^T (c_1 \underset{\sim}{u}_1 + \cdots + c_p \underset{\sim}{u}_p)}$$

$$= \frac{\sum_{k=1}^{n} \lambda_k c_k^2}{\sum_{j=1}^{n} c_j^2}$$

$$= \sum_{k=1}^{n} \lambda_k w_k \quad , \quad w_k = \frac{c_k^2}{\sum_{j=1}^{n} c_j^2}, \quad k = 1, \ldots, n.$$

Now consider minimizing or maximizing $\sum_{k=1}^{n} \lambda_k w_k$ subject to

$$w_1, \ldots, w_n \in [0,1] \quad \text{and} \quad \sum_{k=1}^{n} w_k = 1.$$

Since $\lambda_1 \geq \cdots \geq \lambda_n$, the min and max are $\lambda_n$ and $\lambda_1$, resp.

**9** Let $(X_1, X_2, X_3)$ have covariance matrix $\Sigma = (1-\tau)I_3 + \tau \underset{\sim}{1}_3 \underset{\sim}{1}_3^T$.

(a) Show that $1-\tau$ and $1+2\tau$ are eigenvalues of $\Sigma$.

Can show that

$$\begin{vmatrix} 1-\lambda & \tau & \tau \\ \tau & 1-\lambda & \tau \\ \tau & \tau & 1-\lambda \end{vmatrix} = \cdots = a^3 - 3a\tau^2 + 2\tau^3 \quad \text{with} \quad a = 1-\lambda.$$

$$= (a-\tau)^2 (a+2\tau) \quad \text{with some algebra!}$$

From here we see that the roots of the characteristic equation are $a = \tau$ and $a = -2\tau$, so

$$1-\lambda = \tau \implies \lambda = 1-\tau$$

$$1-\lambda = -2\tau \implies \lambda = 1+2\lambda.$$

(b) Need $-\frac{1}{2} < \tau < 1$ to make all eigenvalues positive.

(c) We have $\mathrm{Var}\left(v_1 X_1 + v_2 X_2 + v_3 X_3\right) = \underset{\sim}{v}^T \Sigma \underset{\sim}{v}$, with $\underset{\sim}{v} = (v_1, v_2, v_3)^T$.

Since $\inf\limits_{\|\underset{\sim}{v}\|_2 = 1} \underset{\sim}{v}^T \Sigma \underset{\sim}{v} = \lambda_1 = \Lambda_{min}(\Sigma)$, take $\underset{\sim}{v}$ as a corresponding eigenvector.

Find a solution to $(\Sigma - \lambda_1 I)\underset{\sim}{x}_1 = \underset{\sim}{0}$, where $\lambda_1 = 1-\tau$. We have

$$\left[\begin{array}{ccc|c} 1-\lambda_1 & \tau & \tau & 0 \\ \tau & 1-\lambda_1 & \tau & 0 \\ \tau & \tau & 1-\lambda_1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} \tau & \tau & \tau & 0 \\ \tau & \tau & \tau & 0 \\ \tau & \tau & \tau & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -x_2 - x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

So the eigenspace corresponding to $\lambda_1 = 1-\tau$ is $\mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Choose a vector in this space & scale it to have unit norm: $\underset{\sim}{v} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.