$\boxed{1}$ Let P be the matrix of an orthogonal projection onto a subspace V.

(a) Show that P is symmetric.

(b) Show that $\|P\underset{\sim}{v}\| \le \|\underset{\sim}{v}\|$ for any $\underset{\sim}{v}$.

__Solution__:

(a) Since P is an orthogonal projection, we have

$$(P\underset{\sim}{v}) \cdot ((I-P)\underset{\sim}{u}) = 0$$

for any vectors $\underset{\sim}{v}$ and $\underset{\sim}{u}$.

Now, for any vectors $\underset{\sim}{v}$ and $\underset{\sim}{u}$ we have

$$\begin{aligned}
\underset{\sim}{u}\cdot(P\underset{\sim}{v}) &= [P\underset{\sim}{u} + (I-P)\underset{\sim}{u}]\,[\underset{\sim}{v} - (I-P)\underset{\sim}{v}] \\
&= (P\underset{\sim}{u})\cdot\underset{\sim}{v} + \underbrace{(P\underset{\sim}{u})\cdot((I-P)\underset{\sim}{v})}_{=0} + (I-P)\underset{\sim}{u}\cdot[\underbrace{\underset{\sim}{v} - (I-P)\underset{\sim}{v}}_{P\underset{\sim}{v}}] \\
&= (P\underset{\sim}{u})\cdot\underset{\sim}{v} + \underbrace{(I-P)\underset{\sim}{u}\cdot P\underset{\sim}{v}}_{=0} \\
&= (P\underset{\sim}{u})\cdot\underset{\sim}{v}.
\end{aligned}$$

This gives

$$\underset{\sim}{u}^T P\underset{\sim}{v} = \underset{\sim}{u}^T P^T\underset{\sim}{v}$$

for all $\underset{\sim}{u}, \underset{\sim}{v}$, which implies $P = P^T$.

(b) We have

$$\|\underset{\sim}{v}\|^2 = \|P\underset{\sim}{v} + (I-P)\underset{\sim}{v}\|^2 = \|P\underset{\sim}{v}\|^2 + \|(I-P)\underset{\sim}{v}\|^2,$$

since $P\underset{\sim}{v}\cdot(I-P)\underset{\sim}{v} = 0$.

Since $\|(I-P)\underset{\sim}{v}\| \geq 0$, we have the result.

[2] Let A be $m \times n$ with rank r and reduced SVD $A = U_r D V_r^T$.
Then

$$A \; V_r D^{-1} U_r^T A = U_r D V_r^T V_r D^{-1} U_r^T U_r D V_r^T$$
$$= U_r D V_r^T$$
$$= A,$$

where we have used the fact that $U_r^T U_r = I_r$ and $V_r^T V_r = I_r$.

[3] **Claim:** Let P be a symmetric projection matrix. Then rank P = tr P.
Proof:

Since P is a projection matrix, it is idempotent.

Idempotent matrices have eigenvalues equal to 0 or 1 (showed on earlier hw).

Since P is symmetric, the multiplicity of 0 as a solution to the characteristic equation gives the dimension of Nul P. This is given by the Spectral Decomposition Theorem.

The Rank Theorem gives rank P + dim Nul P = n, where n is the number of columns of P as well as the number of eigenvalues, counting multiplicities.

Therefore, rank P is equal to the number of nonzero eigenvalues of P.

Since all the nonzero eigenvalues of P are equal to 1, the rank of P is equal to the sum of the eigenvalues.

Since the trace of a matrix is equal to the sum of its eigenvalues, we have rank P = tr P.

14 (a) Let $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $W = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \\ -1/3 & -1/3 \end{bmatrix}$.

We see that $\text{Col } W \neq \text{Col } X$:

We cannot, for example, construct the column $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of X as a linear combination of the columns of W.

We see this by finding that there is no solution to $X\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

$$\left[\begin{array}{cc|c} 2/3 & -1/3 & 1 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 3 \\ -1 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 3 & 3 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 3 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

=> no solution

(b) Let $\tilde{X} = [\underset{\sim}{1}_n \ X]$ and $\tilde{W} = [\underset{\sim}{1} \ (I - \frac{1}{n}\underset{\sim}{1}_n\underset{\sim}{1}_n^T)X]$.

Then we may write

$$\tilde{X} = [\underset{\sim}{1}_n \ \underset{\sim}{x}_1 \cdots \underset{\sim}{x}_p] \quad \text{and} \quad \tilde{W} = [\underset{\sim}{1}_n \ \underset{\sim}{x}_1^c \cdots \underset{\sim}{x}_p^c],$$

where $\underset{\sim}{x}_j^c$ is column j of X centered so that the column has mean 0.

Now we ask whether

$$\text{Col}[\underset{\sim}{1}_n \ \underset{\sim}{x}_1 \cdots \underset{\sim}{x}_p] = \text{Col}[\underset{\sim}{1}_n \ \underset{\sim}{x}_j^c \cdots \underset{\sim}{x}_p^c].$$

It will be convenient to define $\bar{x}_j = \frac{1}{n}\underset{\sim}{1}_n^T\underset{\sim}{x}_j$ for $j = 1, \dots, p$.

Let $\underset{\sim}{v} \in \text{Col } \tilde{X}$

Then $\exists \ \underset{\sim}{a} \in \mathbb{R}^{p+1}$ such that

$$\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{1}_n a_0 + \sum_{j=1}^p \underset{\sim}{x}_j a_j. \\ &= \underset{\sim}{1}_n a_0 + \sum_{j=1}^p (\underset{\sim}{x}_j - \underset{\sim}{1}_n\bar{x}_j) a_j + \sum_{j=1}^p \underset{\sim}{1}_n \bar{x}_j a_j \\ &= \underset{\sim}{1}_n \left(a_0 + \sum_{j=1}^p \bar{x}_j a_j\right) + \sum_{j=1}^p \underset{\sim}{x}_j^c a_j \\ &\in \text{Col } \tilde{W}. \end{aligned}$$

So $\operatorname{Col}\tilde{X} \subset \operatorname{Col}\tilde{W}$.

Now let $\underset{\sim}{v} \in \operatorname{Col}\tilde{W}$. Then $\exists$ $\underset{\sim}{a} \in \mathbb{R}^{p+1}$ such that

$$\begin{aligned}
\underset{\sim}{v} &= \underset{\sim}{1}_n a_0 + \sum_{j=1}^{p} \underset{\sim}{x}_j^c a_j \\
&= \underset{\sim}{1}_n a_0 + \sum_{j=1}^{p} \left(\underset{\sim}{x}_j - \underset{\sim}{1}_n \bar{x}_j\right) a_j \\
&= \underset{\sim}{1}_n \left(a_0 - \sum_{j=1}^{p} \bar{x}_j a_j\right) + \sum_{j=1}^{p} \underset{\sim}{x}_j a_j \\
&\in \operatorname{Col}\tilde{X}.
\end{aligned}$$

So $\operatorname{Col}\tilde{W} \subset \operatorname{Col}\tilde{X}$.

This gives $\operatorname{Col}\tilde{X} = \operatorname{Col}\tilde{W}$.

$\boxed{5}$ We can take P to be a projection matrix onto $\operatorname{Col} X^T$.

We know that AA^- is a projection onto $\operatorname{Col} A$.

So $X^T(X^T)^-$ is a projection onto $\operatorname{Col} X^T$, so $\underline{\underline{P = X^T(X^T)^-}}$ works.

In particular, we can use $X(X^TX)^-$ as the generalized inverse of X^T.

So we may take

$$\underline{\underline{P = X^TX(X^TX)^-}}.$$