STAT 714 hw 5

Gauss-Markov model, estimation, projections, miscellaneous results

Do problem 4.26 from Monahan. In addition:

1. For some matrices $\mathbf{X}$, $\mathbf{A}$, and $\mathbf{B}$:
   (a) Show that $\mathbf{A}\mathbf{X}^T\mathbf{X} = \mathbf{B}\mathbf{X}^T\mathbf{X}$ if and only if $\mathbf{A}\mathbf{X}^T = \mathbf{B}\mathbf{X}^T$.
   (b) Let $(\mathbf{X}^T\mathbf{X})^-$ be a generalized inverse of $\mathbf{X}^T\mathbf{X}$. Show that $\mathbf{X}(\mathbf{X}^T\mathbf{X})^-$ is a gen. inverse of $\mathbf{X}^T$.
2. Let $Y_i = \mu + \sum_{j=1}^d \xi_{ij}\beta_j + \varepsilon_i$ for $i = 1, \dots, n$ and assume the matrix $(\xi_{ij})_{1\le i\le n, 1\le j\le d}$ has full-column rank. Moreover, suppose $\sum_{j=1}^d \xi_{ij} = 1$ for each $i = 1, \dots, n$.
   (a) Give a representation of the model as $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$.
   (b) Check whether μ is estimable.
   (c) Give the matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{b} = \mathbf{0}$ imposes the constraint $\sum_{j=1}^d \sum_{i=1}^n \xi_{ij}\beta_j = 0$.
   (d) Give the least-squares estimator $\hat{\mu}$ under the constraint in (c) in terms of $Y_1, \dots, Y_n$.
3. Let $\mathbf{X}$ be an $n \times p$ design matrix with the first column a column of ones and $\mathbf{y}$ be an $n \times 1$ vector of responses (which do not all take the same value). Let $\mathbf{P_X}$ be the orthogonal projection onto $\operatorname{Col}\mathbf{X}$ and $\mathbf{P_1}$ be the orthogonal projection onto $\operatorname{Span}\{\mathbf{1}_n\}$.
   (a) Give an interpretation to the quantity $\frac{\|(\mathbf{P_X}-\mathbf{P_1})\mathbf{y}\|^2}{\|(\mathbf{I}-\mathbf{P_1})\mathbf{y}\|^2}$.
   (b) Give the range of values which the quantity in part (a) can take.
   (c) Write down the quantity in part (a) in terms of Y_i, $\hat{Y}_i$, and $\bar{Y}_n$, where the Y_i are the entries of $\mathbf{y}$, $\hat{Y}_i$ are the entries of $\mathbf{P_X}\mathbf{y}$, and $\bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$.
4. Let $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, where $\mathbb{E}\mathbf{e} = \mathbf{0}$ and $\operatorname{Cov}\mathbf{e} = \sigma^2\mathbf{I}_n$, and partition $\mathbf{X}$ as $\mathbf{X} = [\mathbf{x}_1\ \mathbf{X}_2]$, where $\mathbf{x}_1$ is a vector and $\mathbf{X}_2$ is a matrix with one column a column of ones. Suppose $\mathbf{X}$ has full column rank and let $\hat{\mathbf{b}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$. Partition $\hat{\mathbf{b}}$ as $\hat{\mathbf{b}} = [\hat{b}_1\ \hat{\mathbf{b}}_2^T]^T$.
   (a) Show that
   $$\operatorname{Var}\hat{b}_1 = \frac{\sigma^2}{1 - R_1^2}\frac{1}{\|(\mathbf{I}-\mathbf{P_1})\mathbf{x}_1\|^2},$$
   where $R_1^2 = \|(\mathbf{P}_{\mathbf{X}_2} - \mathbf{P_1})\mathbf{x}_1\|^2/\|(\mathbf{I}-\mathbf{P_1})\mathbf{x}_1\|^2$.
   (b) Give an interpretation of the result in part (a).
5. Let $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, where $\mathbb{E}\mathbf{e} = \mathbf{0}$ and $\operatorname{Cov}\mathbf{e} = \sigma^2\mathbf{I}$. Assume the eigenvalues of $\mathbf{X}^T\mathbf{X}$ are all positive.
   (a) Show that there is a unique solution to $\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} = \mathbf{X}^T\mathbf{y}$.
   (b) Show that the mean squared error $\mathbb{E}\|\hat{\mathbf{b}} - \mathbf{b}\|^2$ of $\hat{\mathbf{b}}$ is given by $\sigma^2\sum_{i=1}^p \lambda_j^{-1}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{X}^T\mathbf{X}$.
6. Show under the Aitken model that a contrast $\mathbf{c}^T\mathbf{b}$ is estimable if and only if $\mathbf{c} \in \operatorname{Col}\mathbf{X}^T$.
7. Let $\mathbf{X} = [\mathbf{x}_1\ \cdots\ \mathbf{x}_p]$ be matrix with full column rank and with unit columns. Find the unit vector $\mathbf{a}$ and the scalar A such that $\mathbf{a}^T\mathbf{x}_j = A$ for all $j = 1, \dots, p$. The vector $\mathbf{a}$ is "equiangular" with every column of $\mathbf{X}$ with angle equal to $\cos^{-1}(A)$.