STAT 714 fa 2023

Linear algebra review 2/6

The equation $\mathbf{Ax} = \mathbf{b}$

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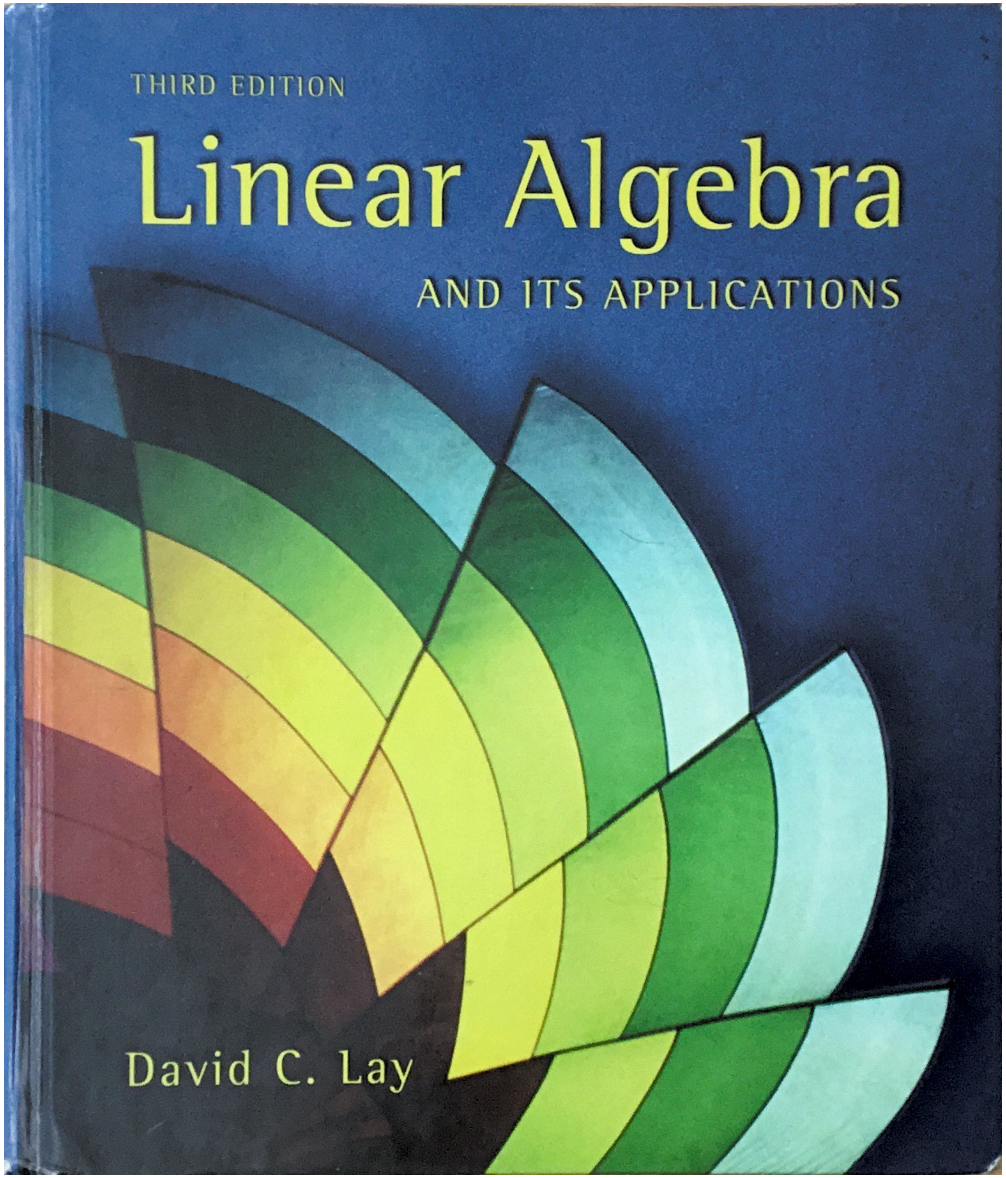
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



1. The equation $\mathbf{A}\mathbf{x} = \mathbf{b}$
2. Elementary row operations and reduced row echelon form
3. Linear independence
4. Finding a matrix inverse with elementary row operations

Example problem: Give solution or characterize solutions if solvable...

$$\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\tilde{x}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}_{\tilde{b}}$$

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 5x_3 &= -3 \end{aligned}$$

The equation $\mathbf{Ax} = \mathbf{b}$

We are often concerned with characterizing the solutions to $\mathbf{Ax} = \mathbf{b}$. It has either

- no solution,
- exactly one solution, or
- infinitely many solutions.

The equation $\mathbf{Ax} = \mathbf{b}$ is called *consistent* if at least one solution exists.

Homogeneous equation

$\underset{\sim}{b}\,2$

A set of linear equations is called *homogeneous* if it can be written as $\mathbf{A}\mathbf{x} = \mathbf{0}$.

To which:

- The solution $\mathbf{x} = \mathbf{0}$ is called the *trivial solution*.
- A nonzero solution is called a *nontrivial solution*.

Example problem: Characterize solution(s) to $\mathbf{A}\mathbf{x} = \mathbf{b}$ if it is consistent, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}.$$

How? Use EROs to put *augmented matrix* $[\underset{\sim}{\mathbf{A}} \;\; \underset{\sim}{\mathbf{b}}]$ in $\boxed{\text{RREF}\ldots}$

1. The equation $\mathbf{Ax} = \mathbf{b}$

2. Elementary row operations and reduced row echelon form

3. Linear independence

4. Finding a matrix inverse with elementary row operations

Elementary row operations

1. Add to one row the multiple of another.
2. Interchange two rows.
3. Multiply a row by a scalar.

These will not change the set of solutions to a linear system of equations.

Example: Consider performing EROs on the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 5x_3 &= -3 \end{aligned}$$

Use elementary row operations to put $[\mathbf{A}\ \ \mathbf{b}]$ in reduced row echelon form...

Reduced row echelon form

A matrix is in *row echelon form* if:

1. All nonzero rows are above all rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry in the row above it.
3. All entries in a column below a leading entry are zeros.

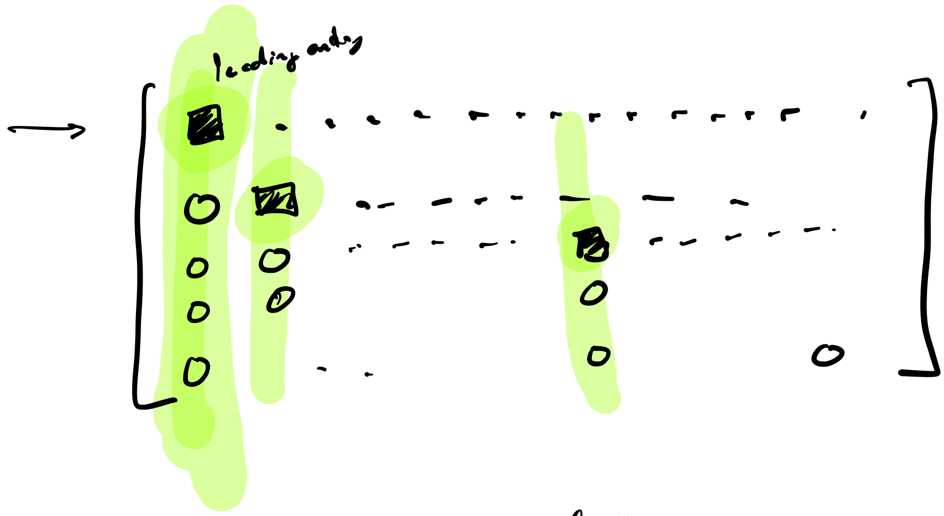
A matrix is in *reduced row echelon form* if in addition to the above:

4. The leading entry of each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

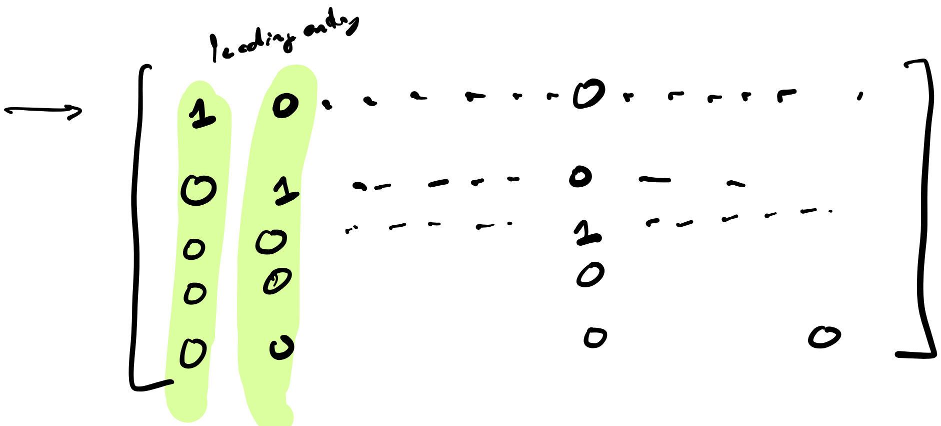
Pivot position/column of a matrix

- A *pivot position* is a location in $\mathbf{A}$ which corresponds to the location of a leading 1 in a row echelon form of $\mathbf{A}$.
- A *pivot column* is a column of $\mathbf{A}$ containing a pivot position.

Use EROs to put $[A \ \underset{\sim}{b}]$ into Echelon form



Row-reduced Echelon Form



Exercise: Put in RREF via EROs the augmented matrix corresponding to

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 5x_3 &= -3 \end{aligned}$$

What is the solution?

$$\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}_{\underset{\sim}{b}}$$

Augmented:

$$[A \quad \underset{\sim}{b}] = \begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = -2$

$x_2 = 1$

$x_3 = 0$

$$\underset{\sim}{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Theorem (RREF and existence of solution to $\mathbf{Ax} = \mathbf{b}$)

1. *Each matrix is row-equivalent to exactly one reduced row echelon matrix.*
2. *An equation $\mathbf{Ax} = \mathbf{b}$ is consistent iff an echelon form of $[\mathbf{A}\ \mathbf{b}]$ has no row like*

$$[0 \ \cdots \ 0 \ b] \quad \textit{with } b \textit{ nonzero.}$$

Recipe for characterizing solutions when $\mathbf{Ax} = \mathbf{b}$ is consistent:

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Exercises: For each system, give solution or characterize solutions if consistent.

1. $$\boxed{\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 1 \\ -2x_2 + x_3 &= 0 \\ 4x_2 - 2x_3 &= 2 \end{aligned}}$$

   No solution

2. $$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ x_2 - x_3 &= 1 \\ x_1 \qquad + x_3 &= 1 \end{aligned}$$

$$
\begin{aligned}
2x_1 + 2x_2 - 3x_3 &= 1 \\
-2x_2 + x_3 &= 0 \\
4x_2 - 2x_3 &= 2
\end{aligned}
$$

Not consistent.

$A\underset{\sim}{x} = \underset{\sim}{b}$

$$
\begin{bmatrix} 2 & 2 & -3 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}
$$

$$
[A \ \underset{\sim}{b}] = \begin{bmatrix} 2 & 2 & -3 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 4 & -2 & 2 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 2 & 2 & -3 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \leftarrow \text{Means NO solution.}
$$

$$
\begin{aligned}
2x_1 + x_2 + x_3 &= 3 \\
x_2 - x_3 &= 1 \\
x_1 \qquad + x_3 &= 1
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}
$$

$$
[A \ \underset{\sim}{b}] = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
$$

$$
\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\begin{aligned}
x_1 \quad + x_3 &= 1 \\
x_2 - x_3 &= 1
\end{aligned}
$$

x_3 is "free"

$x_1 = 1 - x_3$

$x_2 = 1 + x_3$

$x_3 = x_3$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \ x_3 \in \mathbb{R} \right\}
$$

homogeneous equation

$[\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_p]\,\underset{\sim}{x} = \underset{\sim}{0}$

$\boxed{A\underset{\sim}{x} = \underset{\sim}{0}}$

Linear independence of a set of vectors in $\mathbb{R}^n$

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set is

- *linearly independent* if $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution.
- *linearly dependent* if $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ for some $c_1, \dots, c_p$ not all zero.

Exercise: Check whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$V\underset{\sim}{x} = \underset{\sim}{0}$

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix} \rightarrow$ eventually you find unique sol.

$\underset{\sim}{v}_p = \underset{\sim}{0}$. Then $\{\underset{\sim}{v}_1 \ldots \underset{\sim}{v}_p\}$ is lin. dep. because

$c_1 \underset{\sim}{v}_1 + \cdots + c_k \underset{\sim}{v}_k + \cdots + c_p \underset{\sim}{v}_p = 0$

$c_k \neq 0$

$c_j = 0 \quad j \neq k$.

Theorem (Characterization of linearly dependent sets)

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$, where $j > 1$, is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Prove the result.

Suppose $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p\}$ is lin. dep.

Then $c_1 \underset{\sim}{v}_1 + \cdots + c_p \underset{\sim}{v}_p = \underset{\sim}{0}$ for $c_1, \ldots, c_p$ not all zero.

If $c_p \neq 0$ then $\underset{\sim}{v}_p = -\left(\frac{c_1}{c_p}\right)\underset{\sim}{v}_1 - \cdots - \left(\frac{c_{p-1}}{c_p}\right)\underset{\sim}{v}_{p-1}$

non zero

$\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ $\quad\quad$ $\underset{\sim}{v}_1 = 0$

$c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 = \underset{\sim}{0}$ for c_1, c_2 not both zero.

$$\underset{\sim}{v}_2 = -\frac{c_1}{c_2} \underset{\sim}{v}_1$$

X

$n \times p$

$p > n$

Theorem (When the number of vectors exceeds the dimension)

Any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *in* $\mathbb{R}^n$ *is linearly dependent if* $p > n$.

Prove the result.

p variables

n equations

p unknowns

$$n \left\{ \begin{bmatrix} \underset{\sim}{v}_1 & \cdots & \underset{\sim}{v}_p \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0} \right.$$

Does this have only the solution $\underset{\sim}{x} = \underset{\sim}{0}$?

If so, then yes $\underset{\sim}{v}_1 \ldots \underset{\sim}{v}_p$ are lin. indep.

$$\begin{bmatrix} \underset{\sim}{v}_1 & \cdots & \underset{\sim}{v}_p & \underset{\sim}{0} \end{bmatrix}$$

$$n \left\{ \left[\begin{array}{ccc|c} v_{11} & & v_{p1} & 0 \\ \vdots & & & \\ v_{1n} & & v_{pn} & 0 \end{array} \right] \right.$$

p

need to have p pivot columns to have $\underset{\sim}{x} = 0$ be the only solution.

If $n < p$, I cannot get enough pivot cols.

1. The equation $\mathbf{Ax} = \mathbf{b}$

2. Elementary row operations and reduced row echelon form

3. Linear independence

4. Finding a matrix inverse with elementary row operations

Theorem (Finding the inverse using EROs)

An $n \times n$ matrix $\mathbf{A}$ is invertible iff if $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$. In this case any sequence of EROs that reduces $\mathbf{A}$ to $\mathbf{I}_n$ transforms $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Each ERO is equivalent to premultiplication by an *elementary matrix*.

Since EROs can be undone, elementary matrices are invertible.

Exercise: Write down the elementary matrix for the ERO "add to the second row three times the first row." Write down its inverse.

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

Exercise: Find (provided it exists) the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix. (invertible = nonsingular)

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set. (full column rank)

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

$$\underbrace{A(A^{-1}}_{AA^{-1}=I}\underset{\sim}{b}) = \underset{\sim}{b}.$$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

Theorem (Invertibility of $\mathbf{A}$ and the solution to $\boxed{\mathbf{A}}\mathbf{x} = \mathbf{b}$)

If $\mathbf{A}$ is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$ the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Exercise: Find the solution to $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ using the above result.

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{1(3)-(2)(-2)} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

unique solution.

$$\underset{\sim}{x} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.